Exercise Sheet – Combination of Theories

Exercise 1
Prove that the following combination of Linear Arithmetic over $\mathbb{R}$ and Uninterpreted Functions is satisfiable using the Nelson–Oppen procedure:

$$(x_2 \geq 0) \land (f(x_1) = x_3) \land (f(x_1) \geq x_1) \land (x_1 - x_2 \geq x_3) \land (f(x_3) \geq 0)$$

a) Show the formula after purification with optimization, where the same functions or values are assigned to the same auxiliary variable.

b) Show the equalities implied by the purified formula and follow the rest of the Nelson-Oppen procedure until no further equality can be implied.

Exercise 2
Prove that the following combination of three different theories (Linear Arithmetic over $\mathbb{R}$, Uninterpreted Functions starting with the letters a-m and Uninterpreted Functions starting with the letters n-z) is unsatisfiable using the Nelson-Oppen procedure. Therefore, use a table with three columns ($F_1,F_2,F_3$), one per theory.

$$(x_2 \geq x_1) \land (x_1 - x_3 \geq x_2) \land (z(f(x_1) - f(x_2)) \neq z(x_3)) \land (x_3 \geq 0)$$

Exercise 3
Prove that the following combination of Linear Arithmetic over $\mathbb{Z}$ and Uninterpreted Function is unsatisfiable using the Nelson-Oppen procedure for non-convex formulas.

$$1 \leq x \land x \leq 2 \land f(x) \neq f(1) \land f(x) \neq f(2)$$

Exercise 4
Prove that the following combination of Linear Arithmetic over $\mathbb{Z}$ and Uninterpreted Function is unsatisfiable using the Nelson-Oppen procedure for non-convex formulas.

$$(1 \leq x_1) \land (x_2 \leq 3) \land (x_3 \leq x_2) \land (x_1 \leq 3 - x_3) \land (x_3 = 1) \land g(x_1) \land \neg g(2) \land f(x_2) \neq f(x_1) \land f(x_2) \neq f(3) \land \neg g(x_2 - x_1)$$