Exercise 1 (Progress Property)

Let \( t \) be a closed and well-typed term, i.e. \( \vdash t : \tau \) for some \( \tau \). Show that \( t \) is either a value or there is a \( t' \) such that \( t \rightarrow_{\text{cbv}} t' \).

Exercise 2 (Normal Form)

Show that every type-correct \( \lambda \rightarrow \)-term has a \( \beta \)-normal form.

Homework 3 (Typing)

a) Prove:

\[
\vdash (\lambda x : \tau_2 \rightarrow \tau_3. \lambda y : \tau_1 \rightarrow \tau_2. \lambda z : \tau_1. x (y z)) : (\tau_2 \rightarrow \tau_3) \rightarrow (\tau_1 \rightarrow \tau_2) \rightarrow \tau_1 \rightarrow \tau_3
\]

b) Give suitable solutions for \( \tau_1, \tau_2, \tau_3 \) and \( \tau_4 \) and prove that the term is type-correct given your solution.

\[
\vdash \lambda x : ? \tau_1. \lambda y : ? \tau_2. \lambda z : ? \tau_3. x \ y \ z : ? \tau_4
\]

Homework 4 (\( \beta \)-reduction preserves types)

A type system has the subject reduction property if evaluating an expression preserves its type. Prove that the simply typed \( \lambda \)-calculus (\( \lambda \rightarrow \)) has the subject reduction property:

\[
\Gamma \vdash t : \tau \land t \rightarrow_{\beta} t' \Longrightarrow \Gamma \vdash t' : \tau
\]

Hints: Use induction over the inductive definition of \( \rightarrow_{\beta} \) (Def. 1.2.2). State your inductive hypotheses precisely – it may help to introduce a binary predicate \( P(t, t') \) to express the property you are proving by induction. Also note that the proof will require rule inversion: Given \( \Gamma \vdash t : \tau \), the shape of \( t \) (variable, application, or \( \lambda \)-abstraction) may determine which typing rule must have been used to derive the typing judgment.

Within your proof, you are free to use the following lemma about substitution:

\[
\Gamma \vdash u : \tau_0 \land \Gamma [x : \tau_0] \vdash t : \tau \Longrightarrow \Gamma \vdash t[u/x] : \tau
\] (1)