Exercise 1 (Peirce’s Law in Intuitionistic Logic)

Prove the following variant of Peirce’s Law in intuitionistic logic:

\[((((P \to Q) \to P) \to P) \to Q) \to Q\]

Exercise 2 (Intuitionistic Proof Search in Haskell)

The goal of this exercise is to implement the procedure to decide $\Gamma \vdash A$ in Haskell, i.e. the algorithm from the proof of Theorem 4.0.6.

- Have a look at the template provided on the website. It provides definitions of formulae and proof terms of intuitionistic propositional logic.
- Try to fill in the implementation of `solve`.
- Implement the three proof rules seen in the lecture: assumption, intro, and elim. Use the examples at the end of the template to test your implementation as you go. For elim, use the criterion from the proof to guess suitable instantiations.

Homework 3 (Constructive Logic)

a) Prove the following statement using the calculus for intuitionistic propositional logic:

\[((c \to b) \to b) \to (c \to a) \to ((a \to b) \to b)\]

*Hint:* To make your proof tree more compact, you may remove unneeded assumptions to the left of the $\vdash$ during the proof as you see fit. For example, the following step is valid:

\[
\frac{p, q \vdash p}{p \vdash p}
\]

b) Give a well-typed expression in $\lambda^+$ with the type

\[((\gamma \to \beta) \to \gamma) \to (\gamma \to \alpha) \to ((\alpha \to \beta) \to \beta)\]

(You don’t need to give the derivation tree.)
Homework 4 (The Negative Fragment)

We say that a formula $A$ is negative if atomic formulas $P$ only occur negated in $A$, i.e. in the form $P \rightarrow \bot$ ($\neg P$ for short). The symbol $\bot$ for falsehood plays the role of an unprovable propositional constant: we do not have any special proof rules or axioms for it.

Show that if $A$ is negative, then:

$$\vdash \neg\neg A \rightarrow A$$

**Hint:** First show:

a) $\vdash \neg\neg\neg A \rightarrow \neg A$

b) $\vdash \neg\neg(A \rightarrow B) \rightarrow (\neg\neg A \rightarrow \neg\neg B)$

c) $\vdash (\neg\neg A \rightarrow \neg\neg B) \rightarrow (A \rightarrow \neg\neg B)$