Exercise 1 (Confluence & Commutation)

Show: If $\rightarrow_1$ and $\rightarrow_2$ are confluent, and if $\rightarrow^*_1$ and $\rightarrow^*_2$ commute, then $\rightarrow_{12} := \rightarrow_1 \cup \rightarrow_2$ is also confluent.

Solution

Lemma A.3.2 from the lecture. The key idea is to consider $\rightarrow^*_1 \circ \rightarrow^*_2$ as $\rightarrow^*_{12}$ unfolds into iterations of this relation. More precisely:

$$\rightarrow_{12} \subseteq \rightarrow^*_1 \circ \rightarrow^*_2 \subseteq \rightarrow^*_{12} \quad (*)$$

It is easy to see that $\rightarrow^*_1 \circ \rightarrow^*_2$ has the diamond property (see picture in the lecture notes). Thus $\rightarrow^*_1 \circ \rightarrow^*_2$ is strongly confluent, and together with the “sandwich” property for $\rightarrow^*_1 \circ \rightarrow^*_2$ and $\rightarrow_{12}$, we get that $\rightarrow_{12}$ is confluent.

Exercise 2 (Local Commutation)

Show: If $t_2 \leftarrow s \rightarrow t_1 = \exists u. t_2 \rightarrow^*_1 u \leftarrow t_1$, then $\rightarrow^*_1$ and $\rightarrow^*_2$ commute.

Here $\rightarrow^*_1$ denotes the reflexive closure of $\rightarrow$, i.e.:

$$\rightarrow^*_1 := \rightarrow \cup \rightarrow^0$$

Solution

Lemma A.3.3 from the lecture.

Exercise 3 (Strong Confluence)

A relation $\rightarrow$ is said to be strongly confluent iff:

$$t_2 \leftarrow s \rightarrow t_1 = \exists u. t_2 \rightarrow^* u \leftarrow t_1$$

Show that every strongly confluent relation is also confluent.
Solution

We show that every strongly confluent relation is also semi-confluent (see homework). To do so, we will show the stronger property

\[ t_2 \xleftarrow{n} s \rightarrow t_1 \implies \exists u. t_2 \rightarrow^* u \xleftarrow{*} t_1 \]

by induction on \( n \). The base case for \( n = 0 \) is trivial (choose \( u = t_1 \)). For the induction step, we assume the statement for some \( u \) as the induction hypothesis and assume another step \( t_2 \rightarrow t'_2 \). We make a case distinction on \( t_2 \rightarrow u \). The case \( t_2 = u \) is trivial as then \( s \rightarrow^{n+1} t'_2 \). If \( t_2 \rightarrow u \), then from strong confluence we obtain a \( u' \) such that

\[ u \rightarrow^* u' \land t'_2 \rightarrow^* u' \]

Together with the induction hypothesis, this concludes the proof.

Homework 4 (Semi-Confluence)

A relation \( \rightarrow \) is said to be semi-confluent iff:

\[ t_2 \rightarrow^* s \rightarrow t_1 \implies \exists u. t_2 \rightarrow^* u \rightarrow t_1 \]

Show that \( \rightarrow \) is semi-confluent if and only if it is confluent.

Homework 5 (Diamond Property & Normal Forms)

Show that if \( \rightarrow \) has the diamond property, every element is either in normal form or has no normal form.

Homework 6 (Weak Diamond Property)

Assume that \( \rightarrow \) has the following weaker diamond property:

\[ t_2 \xleftarrow{s} t_1 \land t_1 \neq t_2 \implies \exists u. t_2 \rightarrow u \xleftarrow{t_1} \]

a) Is it still the case that every element is either in normal form or has no normal form?

b) Show that if \( t \) has a normal form, then all its reductions to its normal form have the same length.