Homework is due on April 27, before the tutorial.

**Exercise 1 (H) (Bounded Relations)**

A relation $\rightarrow$ over the set $A$ is called *bounded*, if for each element $x$, the lengths of all paths from $x$ are bounded. Formally:

$$\forall x \in A. \exists n. \not\exists y \in A. x \stackrel{n}{\rightarrow} y$$

Prove or refute:

a) Each terminating relation is bounded.

b) A finitely branching relation is terminating if and only if it is bounded. (Hint: Well-founded induction)

c) Now we call a relation *globally bounded*, if there is a bound that is valid for all elements. Formally:

$$\exists n. \forall x \in A. \not\exists y \in A. x \stackrel{n}{\rightarrow} y$$

Prove or refute: Any finitely branching and terminating relation is globally bounded.

**Exercise 2 (H) (Partial Ordering)**

Prove or refute:

a) $\rightarrow^+$ is a strict partial order if and only if $\rightarrow$ is acyclic.

b) $\rightarrow^*$ is a partial order if and only if $\rightarrow$ is acyclic.

Notes: A relation $R \subseteq X \times X$ is called *strict partial order* if it is irreflexive ($\forall x \in X. \lnot(x R x)$), transitive ($\forall x, y, z \in X. x R y \land y R z \implies x R z$), and asymmetric ($\forall x, y \in X. x R y \implies \lnot(y R x)$).

A relation $R \subseteq X \times X$ is called *partial order* if it is reflexive ($\forall x \in X. x R x$), transitive and antisymmetric ($\forall x, y \in X. x R y \land y R x \implies x = y$).

**Exercise 3 (T) (Example)**

Let $(\mathbb{N}_+, \rightarrow)$ be the reduction system on positive natural numbers, where $\rightarrow = \{(n, m) \mid 3n = 2m \lor 5n = 7m\}$.

a) Does this system terminate? Justify your answer.

b) Determine the set of all irreducible elements.

c) What is the normal form of 630? Show: $10 \leftrightarrow 14$ and $20 \leftrightarrow^* 63$. 
Exercise 4 (T) (Equivalence Relations)

A relation $R \subseteq X \times X$ is called Equivalence relation, if:

- $R$ is reflexive, i.e. $\forall x \in X. \ x \ R \ x$
- $R$ is transitive, i.e. $\forall x, y, z \in X. \ x \ R \ y \land y \ R \ z \implies x \ R \ z$
- $R$ is symmetric, i.e. $\forall x, y \in X. \ x \ R \ y \implies y \ R \ x$

Let $\rightarrow$ be a relation. Show: $\leftrightarrow$ is the smallest equivalence relation that contains $\rightarrow$.

Exercise 5 (T) (Confluence and Normal Form)

- Show that a reduction system $(A, \rightarrow)$ is confluent and normalizing, if and only if every element has a unique normal form.
- Give an example of a confluent and normalizing reduction system that does not terminate.