Homework is due on May 25, before the tutorial.

Much of this exercise sheet involves the use of Birkhoff’s Theorem, which tells us that various relations induced by \( E \) are equivalent. So for each problem, we are free to choose the most appropriate relation:

\[
s \leftrightarrow_s \; t \iff E \vdash s \approx t \iff E \vdash s \approx t
\]

**Exercise 1 (H) (Equivalence Classes)**

Let \( \Sigma = \{ f, g \} \) and

\[
E = \{ f(f(x)) \approx f(x), \; g(f(x)) \approx f(x), \; f(g(x)) \approx g(x), \; g(g(x)) \approx g(x) \}
\]

a) Describe the equivalence classes of \( \approx_E \).

b) For each equivalence class \([t]_{\approx_E}\), determine a shortest term \( \hat{t} \) in \([t]_{\approx_E}\).

c) Give a model for \( E \) that has more than one element.

**Exercise 2 (H) (Congruence Closure)**

Let \( \Sigma = \{ f, a, b \} \) and \( G = \{ f(a, b) \approx a \} \). Using congruence closure, decide whether the terms \( s \) and \( t \) are equivalent with respect to the equation set \( G \). Use the abstract algorithm on equation sets.

a) \( s = f(f(a, b), b), \quad t = a \)

b) \( s = f(f(a, b), b), \quad t = b \)

**Exercise 3 (H) (More General Substitutions)**

Let \( \sigma \) and \( \sigma' \) be substitutions such that each one is an instance of the other: Formally, we have both \( \sigma \lesssim \sigma' \) and \( \sigma \gtrsim \sigma' \). Show that in this case, there must exist a renaming \( \rho \) (i.e., an injective substitution where \( \text{Ran}(\rho) \subseteq V \)) such that \( \sigma = \rho \sigma' \).

(Continued on back)
Exercise 4 (T) (Consistency)

A set $E$ of equations is called consistent if there exists a model of $E$ with more than one element. Show the following statements:

a) $E$ is inconsistent if and only if $E \vdash x \approx y$ (for distinct variables $x$ and $y$) is derivable as a syntactic consequence of $E$.

b) If $E$ includes an equation of the form $t \approx x$ with $x \notin \text{Var}(t)$, then $E$ is inconsistent.

c) If there is a model of $E$ with two elements, then for each $n \in \mathbb{N}$ there is a model of $E$ with $2^n$ elements. Hint: Consider pairs.

Exercise 5 (T) (Commutativity of $+$)

Let $\Sigma = \{0, s, +\}$ and $E = \{x + 0 \approx x, x + s(y) \approx s(x) + y\}$.

a) Show that $E \models x + s(s(0)) \approx s(s(x))$ and $E \models s^i(0) + s^i(0) \approx s^j(0) + s^j(0)$.

b) Show that $E \not\models x + y \approx y + x$.
   Hint: Give a model of $E$ where $+$ is not commutative.