Exercise 1 (H) (\(\lambda\)-Calculus Interpreter)

Implement an interpreter for the lambda calculus in your favorite programming language. It shall convert a lambda-term into its \(\beta\)-normal form. If there is an infinite sequence of \(\beta\)-reductions, your program is allowed not to terminate. Comment your program thoroughly.

Test your program for (at least) the following examples, and document the results:

\[
\begin{align*}
t_1 &= (\lambda x. (\lambda y. x y)) y \\
t_2 &= (\lambda x. \lambda x. x) y \\
t_3 &= (k c) \text{omega} \\
t_4 &= (\lambda x. ((f x) x)) 5 \\
t_5 &= (\lambda x. x) (\lambda x. x) \\
t_6 &= x (\lambda y. y) \\
t_7 &= \text{self} (\lambda x. (f x)) \\
t_8 &= ((s k) x) y \\
t_9 &= ((s k) k) x \\
t_{10} &= (\lambda x. (\lambda y. (x i)) (\lambda y. (x y))) \\
t_{11} &= (\lambda x. ((\lambda y. (x y)) c)) (i d)
\end{align*}
\]

where

\[
\begin{align*}
\text{self} &= (\lambda x. (x x)) & \text{omega} &= \text{self self} \\
k &= \lambda x. (\lambda y. x) & s &= \lambda x. (\lambda y. (\lambda z. ((x z) (y z)))) \\
i &= \lambda x. x
\end{align*}
\]

Hints:

- This exercise works best in a functional programming language.
- You are not required to write a parser/pretty printer, but may encode the example terms directly in your program.
- \(t_1\) and \(t_2\) test whether you have implemented substitutions right, there normal forms are \(\lambda y'. y y'\) and \(\lambda x. x\).

Exercise 2 (H) (Substitution Lemma)

Show that:

\[
s[t/x][u/y] = s[u/y][t[u/y]/x] \text{ if } x \notin \text{FV}(u)
\]

Exercise 3 (T) (\(\lambda\)-Terms)

Evaluate the following substitutions:

a) \((\lambda y. x (\lambda x. x)) [(\lambda y. x y)/x]\)  

b) \((y (\lambda v. xv)) [(\lambda y. vy)/x]\)

Rewrite the following terms such that they are completely parenthesized and conform to the grammar for the \(\lambda\)-calculus given in the lecture (without any shortcut notations).
c) \(ux(yz)(\lambda v.vy)\)  

d) \((\lambda xyz.xz)(yz)uvw\)

Rewrite the following terms such that there are as few parentheses as possible, and apply all shortcut notation from the lecture:

e) \(((u(\lambda x. (v(wx))))x)\)  

f) \(((w(\lambda x. (\lambda y. (\lambda z. ((xz)(yz))))))u)v)\)

**Exercise 4 (T) (Formalization with \(\lambda\)-Terms)**

Express the following propositions as \(\lambda\)-terms. Use the constant \(D\) as a derivative operator.

a) The derivative of \(x^2\) is \(2x\).

b) The derivative of \(x^2\) at 3 is 6.

c) Let \(f\) be a function, and let \(g\) be defined as \(g(x) := f(x^2)\). The derivative of \(g\) at \(x\) is different from the derivative of \(f\) at \(x^2\).

d) Formulate the proposition c) without using the auxiliary function symbol \(g\).