Homework is due on July 13th, before the tutorial.

Exercise 1 (H) (*Church-Encodings: Trees*)

a) Encode a datatype of binary trees in lambda calculus. Specify the operations \texttt{tip} and \texttt{node} that construct trees, as well as \texttt{isTip}, \texttt{left}, and \texttt{right}. Show that the following holds:

\[
\begin{align*}
\text{isTip} \texttt{tip} & \rightarrow^* \texttt{true} \\
\text{isTip} \texttt{(node x y)} & \rightarrow^* \texttt{false} \\
\text{left} \texttt{(node x y)} & \rightarrow^* x \\
\text{right} \texttt{(node x y)} & \rightarrow^* y
\end{align*}
\]

Exercise 2 (H) (*Parallel $\beta$-Reduction*)

In the lecture, we defined parallel $\beta$-reduction $> \in$ inductively as follows:

\[
\begin{align*}
s > s' & \quad \Rightarrow \quad \lambda x. s > \lambda x. s' \\
s > s' \land t > t' & \quad \Rightarrow \quad (s \ t) > (s' \ t') \\
s > s' \land t > t' & \quad \Rightarrow \quad (\lambda x. s) \ t > s' [t' / x]
\end{align*}
\]

We also showed $> \subseteq \rightarrow^*_\beta$. In this exercise, you have to show: $\rightarrow^*_\beta \subseteq >$

Hint: Use induction over the inductive definition of $\rightarrow^*_\beta$ (Def. 1.2.2).

Exercise 3 (T) (*Lists*)

Specify $\lambda$-Terms for \texttt{nil}, \texttt{cons}, \texttt{hd}, \texttt{tl} and \texttt{null}, that encode lists in the $\lambda$-calculus. Show that your terms satisfy the following:

\[
\begin{align*}
\texttt{null \ nil} & \rightarrow^* \texttt{true} & \texttt{hd \ (cons \ x \ l)} & \rightarrow^* x \\
\texttt{null \ (cons \ x \ l)} & \rightarrow^* \texttt{false} & \texttt{tl \ (cons \ x \ l)} & \rightarrow^* l
\end{align*}
\]

Hint: Use pairs.

Exercise 4 (T) (*Confluence of $\beta$-reduction*)

In the lecture we have shown the confluence of $\rightarrow^*_\beta$ using the diamond property of parallel $\beta$-reduction (cf. Exercise 2). In this exercise, we develop an alternative proof.

We define the operation $*$ on $\lambda$-terms inductively over the structure of terms:

\[
\begin{align*}
x^* & = x \\
(\lambda x. t)^* & = \lambda x. t^* \\
(t_1 t_2)^* & = t_1^* t_2^* \quad \text{if } t_1 t_2 \text{ is not } \beta\text{-reducible.} \\
((\lambda x. t_1) t_2)^* & = t_1^* [t_2 / x]
\end{align*}
\]
a) Show that we have for two arbitrary λ-terms $s$ and $t$: $s > t \implies t > s^*$

b) Show that $\rightarrow_\beta$ is confluent.