Exercise 39 (Critical Pairs)

Specify all critical pairs of the following term rewriting systems:

a) \( f(g(f(x))) \rightarrow x, \ f(g(x)) \rightarrow g(f(x)) \)
b) \( g(f(x)) \rightarrow f(g^2(h(x))), \ h(f(x)) \rightarrow f(g(h^2(x))), \ h(g(x)) \rightarrow g(h(x)) \)

Which systems are locally confluent, which are convergent (i.e., terminating and confluent)?

Solution

a) Three critical pairs:

\[
\begin{array}{c}
\xrightarrow{x} & \xrightarrow{g(x)} \\
\xrightarrow{\overline{f(g(f(x))}}} & \xrightarrow{\overline{f(g(f(x)))}} \\
\xrightarrow{g(f(f(x))}) & \xrightarrow{f(g(f(f(x))))} \\
\xrightarrow{\overline{f(g(g(f(f(x))))})} & \xrightarrow{f(g(x))} & \xrightarrow{} & \xrightarrow{g(f(x))} & \xrightarrow{f(g(x))}
\end{array}
\]

Not locally confluent, not convergent.

b) One critical pair:

\[
\begin{array}{c}
\xrightarrow{\overline{g(h(f(x))}}} & \xrightarrow{f(g(h^2(x))))} & \xrightarrow{f(g^2(h(g(h^2(x))))))} \\
\xrightarrow{\overline{h(g(f(x))}}} & \xrightarrow{g(f(g(h^2(x))))} & \xrightarrow{f(g^2(h^3(x))} & \xrightarrow{f(g^3(h^3(x))}} & \xrightarrow{}
\end{array}
\]

Locally confluent, terminating, hence convergent.

Exercise 40 (Confluence)

Determine terms \( r_1 \) and \( r_2 \) such that \( \{ f(g(x)) \rightarrow r_1, \ g(h(x)) \rightarrow r_2 \} \) is confluent.
Solution

- A trivial solution is \( r_1 = f(g(x)) \) and \( r_2 = g(h(x)) \). This is obviously confluent, but does not terminate.

- An alternative solution is described in the following:
  
  \[ R \text{ has a critical pair:} \]
  
  \[ \begin{array}{c}
  r_1[h(x)] \\
  \downarrow \uparrow \text{?} \downarrow \uparrow t \\
  \nearrow \nwarrow f(r_2[x]) \\
  \end{array} \]

  Thus, \( r_1 \) and \( r_2 \) have to be chosen such that:

  \[ r_1[h(x)] \xrightarrow{*} t \leftarrow f(r_2[x]) \]

  With \( r_1 = g(x) \) and \( r_2 = g(x) \) we get:

  \[ \begin{array}{c}
  g(h(x)) \\
  \downarrow \uparrow f(g(h(x))) \\
  \nearrow \nwarrow g(x) \\
  \downarrow \uparrow f(g(x)) \\
  \end{array} \]

  Hence, \( R \) is confluent.

Exercise 41 (Newman’s Lemma)

Give an indirect proof of Newman’s Lemma, by showing that \( \rightarrow \) has an infinite reduction sequence, if \( \rightarrow \) is locally confluent but not confluent.

Hint: Show that every element with two distinct normal forms has a direct successor with two distinct normal forms.

Solution

Proof. Let \( \rightarrow \) be locally confluent but not confluent. We assume that \( \rightarrow \) terminates and derive a contradiction.

not confluent \( \Rightarrow \) \( \exists x \) with two distinct normal forms \( \text{NF}_1 \) and \( \text{NF}_2 \)

\( \Rightarrow \) \( x \) has at least two direct successors: \( a \leftarrow x \rightarrow b \)

We first show that \( a \) or \( b \) again have two different normal forms. With this, we can construct an infinite descending chain of direct successors with different normal forms.

As \( \rightarrow \) is locally confluent, there exists an \( u \) with \( a \xrightarrow{*} u \xleftarrow{*} b \).
Since \( \rightarrow \) terminates by assumption, \( u \) has at least one normal form. Case distinction:

\[
\begin{align*}
\mathcal{u} \longrightarrow \text{NF}_1 : & \quad b \longrightarrow \text{NF}_2 \land b \longrightarrow u \longrightarrow \text{NF}_1 \\
& \Rightarrow b \text{ has two distinct NFs} \\
\mathcal{u} \longrightarrow \text{NF}_2 : & \quad a \longrightarrow \text{NF}_1 \land a \longrightarrow u \longrightarrow \text{NF}_2 \\
& \Rightarrow a \text{ has two distinct NFs} \\
\mathcal{u} \longrightarrow \text{NF}_3 \land \text{NF}_1 \neq \text{NF}_3 \neq \text{NF}_2 : & \quad \text{both } a \text{ and } b \text{ have distinct NFs}
\end{align*}
\]

\[\square\]

**Homework 42 (Diamond Property)**

Let \( \rightarrow \) be a TRS with the following “diamond property”:

\[
y \leftarrow x \longrightarrow z \land y \neq z \implies \exists u. y \longrightarrow u \leftarrow z
\]

Show that, if \( a \) has a normal form, all reductions from \( a \) to this normal form have the same length.

**Homework 43 (Critical Pairs’)**

Specify all critical pairs of the following term rewriting systems:

\begin{enumerate}
\item a) \( f(x, x) \longrightarrow a, f(x, g(x)) \longrightarrow b \)
\item b) \( f(f(x, y), z) \longrightarrow f(x, f(y, z)), f(x, a) \longrightarrow x \)
\item c) \( f(f(x, y), z) \longrightarrow f(x, f(y, z)), f(a, x) \longrightarrow x \)
\item d) \( 0 + y \longrightarrow y, x + 0 \longrightarrow x, s(x) + y \longrightarrow s(x + y), x + s(y) \longrightarrow s(x + y) \)
\end{enumerate}

Which systems are locally confluent, which are convergent (i.e., terminating and confluent)?

**Homework 44 (Completion)**

Complete

\[
E = \{ f(g(f(x))) \approx x \}
\]

to a convergent term rewriting system.