Exercise 45 (Linear Term Rewriting Systems)

A rewrite rule $l \rightarrow r$ is called left-linear if every variable in $l$ occurs exactly once. Similarly, $l \rightarrow r$ is called right-linear if every variable in $r$ occurs exactly once. A rule is linear if it is both right- and left-linear. We say that a term rewriting system is linear if it contains only linear rules.

Show:

a) Every linear term rewriting system $R$ that has no critical pairs is confluent. Give a self-contained proof; do not simply apply Corollary 6.3.11 from the book!

Hint: Show that $R$ is strongly confluent.

b) If $R$ is a linear term rewriting system, and for every critical pair $(t_1, t_2)$ there exists $t_0$ such that $t_1 \rightarrow t_0 \leftarrow t_2$, then $R$ is confluent.

Hint: Extend the proof of the previous statement.

Solution

a) Proof. We show that for any two rules

1. $l_1 \rightarrow r_1$ (1)
2. $l_2 \rightarrow r_2$ (2)

from $R$, we have:

$x \overset{(1)}{\leftarrow} t \overset{(2)}{\rightarrow} y \Rightarrow \exists z. x \overset{\rightarrow}{\rightarrow} z \overset{\leftarrow}{\leftarrow} y$

Hence, $R$ is strongly confluent, and thus also confluent.

Let $p$ be the application position of rule (1) and $q$ the application position of rule (2) in a term $t$.

We have either $p || q$, $p \geq q$ or $q \geq p$. Case distinction:

- The case $p || q$ is easy.
$q \geq p$. As $R$ has no critical pairs, the position $q$ is below the range that is matched by $l_1$ in $t$.

As $R$ is linear, and in particular, all right and left hand sides contain any variable at most once, the application of a rule $l_1 \rightarrow r_1$ can move the application position of rule $l_2 \rightarrow r_2$ to at most one $q'$.

If $r_1$ does not contain the respective variable, the rule $l_2 \rightarrow r_2$ cannot be applied. But we then have:
b) Additionally to the cases from the previous part, we have to regard the case $p < q$ for a critical pair. By assumption, we have for each critical pair $(t_1, t_2)$ a $t_0$ with $t_1 \xrightarrow{\sim} t_0 \xleftarrow{\sim} t_2$:

\[ p \geq q. \text{ Symmetric.} \]

\[ \square \]
Exercise 46 (λ-Terms)

Evaluate the following substitutions:

a) \((\lambda y.x(\lambda x.x))[\lambda y.x y/ x]\)  
b) \((\lambda v.v x)(\lambda v.v y)/ x\)

Rewrite the following terms such that they are completely parenthesized and conform to the grammar for the λ-calculus given in the lecture (without any shortcut notations).

c) \(u x(y z)(\lambda v.v y)\)  
d) \((\lambda x y z.x z)(\lambda y z)(\lambda z)(\lambda x x)(\lambda y y)uvw\)

Rewrite the following terms such that there are as few parentheses as possible, and apply all shortcut notation from the lecture:

e) \(((u(\lambda x.v)(\lambda x.w)))(\lambda x.x)\)  
f) \(((v(\lambda x.v)(\lambda x.w)))(\lambda x.x)uv\\)

Solution

a) \(\lambda y'(\lambda y.x y)(\lambda x.x)\)  
b) \(y(\lambda x.v)(\lambda y.y)z\)  
c) \(((u x)(y)(\lambda v.v y))\)

Exercise 47 (Formalizations with λ-Terms)

Express the following propositions as λ-terms. Use the constant D as a derivative operator.
a) The derivative of $x^2$ is $2x$.

b) The derivative of $x^2$ at 3 is 6.

c) Let $f$ be a function, and let $g$ be defined as $g(x) := f(x^2)$. The derivative of $g$ at $x$ is different from the derivative of $f$ at $x^2$.

d) Formulate the proposition c) without using the auxiliary function symbol $g$.

Solution

a) $D(\lambda x.x^2) = \lambda x.2x$

b) $(D(\lambda x.x^2)) 3 = 6$

c) $(Dg)x \neq (Df)(x^2)$

d) $(D(\lambda x.f(x^2)))x \neq (Df)(x^2)$

Homework 48 (Strong Confluence)

Let $\rightsquigarrow$ be a relation with $\rightarrow \subseteq \rightsquigarrow \subseteq \ast \rightarrow$.

Show that $\rightsquigarrow$ is strongly confluent iff \( \forall t_1 t_2 s. t_1 \leftarrow s \rightsquigarrow t_2 \implies \exists t. t_1 \rightsquigarrow^* t \leftarrow t_2. \)

(Strong confluence of $\rightarrow$ is \( \forall t_1 t_2 s. t_1 \leftarrow s \rightarrow t_2 \implies \exists t. t_1 \rightarrow^* t \leftarrow t_2. \))

Homework 49 (Confluence)

Let $R$ be the following term rewriting system:

\[
\{ f(x, x) \rightarrow a, \ c \rightarrow g(c), \ g(x) \rightarrow f(x, g(x)) \}\n\]

Is $R$ confluent? Justify your answer.

Homework 50 (Substitution Lemma)

Show that, given $x \neq y$ and $x \notin \text{FV}(u)$:

\[
s[t/x][u/y] = s[u/y][t[u/y]/x] \]