Exercise 7.1.  [QE for DLO]
Use the quantifier-elimination procedure for DLOs to check whether the following formula is a member of \( \text{Th}(DLO) \):

\[
\exists x \forall y \exists z ((x < y \lor z < x) \land y < z)
\]

Exercise 7.2.  [Compactness for FOL]
Prove the Compactness Theorem for first-order logic:

If every finite subset of a (countable) set \( M \) of formulas has a model, then \( M \) as a whole has a model.

Hint: You may use the following definitions and theorems:

Let \( M = \{F_1, F_2, \ldots\} \). Then \( \text{skolem}(M) = \{F'_1, F'_2, \ldots\} \), where \( F'_i \) is a Skolem form of \( F_i \), and the Skolem-functions in the \( F'_i \) are pairwise disjoint.

Note: Wlog, we assume that \( M \) is of a form that allows us to find enough fresh function symbols for Skolem functions.

Theorem: \( M \) is satisfiable iff \( \text{skolem}(M) \) is satisfiable.

Exercise 7.3.  [Axiomatizations and Compactness]
Using compactness, show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if \( \text{Cn}(\Gamma) = \text{Cn}(\Delta) \) with \( \Gamma \) countable and \( \Delta \) finite, then there is a finite \( \Gamma' \subseteq \Gamma \) with \( \text{Cn}(\Gamma') = \text{Cn}(\Gamma) \).
Homework 7.1. [Theories] (5 points)

1. Show $Cn(S) = Th(Mod(S))$, i.e. show $Th(Mod(S)) = \{ F \mid F$ $\Sigma$-sentence and $S \models F \}$

2. Show that $Cn$ is a closure operator, i.e. $Cn$ fulfills the following properties:
   - $S \subseteq Cn(S)$
   - if $S \subseteq S'$ then $Cn(S) \subseteq Cn(S')$
   - $Cn(Cn(S)) = Cn(S)$

Homework 7.2. [Quantifier Elimination for $Th(\mathbb{N}, 0, S, =)$] (5 points)

Give a quantifier-elimination procedure for $Th(\mathbb{N}, 0, S, =)$ where $S$ is the successor operation on natural numbers, i.e. $S(n) = n + 1$.

Hint: $a = b$ iff $S^k(a) = S^k(b)$ for any $a, b, k \in \mathbb{N}$.

Homework 7.3. [Quantifier Elimination for DLOs with endpoints] (5 points)

Let $\Sigma = \{ a, b, <, = \}$ and replace the last two axioms for DLOs with:

- $\forall x (x = a \lor a < x)$
- $\forall x (x = b \lor x < b)$

Modify the quantifier-elimination procedure for dense linear orders to obtain a quantifier-elimination procedure for this theory.

What happens if there is only one endpoint?

Homework 7.4. [Decidable Axiomatizations] (5 points)

Show that any set of sentences that is axiomatized by a recursively enumerable set is also axiomatized by a decidable set.

Hint: For each $n \in \mathbb{N}$ a possible encoding of $n$ as a formula could be of the form

$$F \land \ldots \land F$$

for some formula $F$. 