Exercise 2.1. [Predicate Logic]
a) Specify a satisfiable formula $F$, such that for all models $\mathcal{A}$ of $F$, we have $|U_{\mathcal{A}}| \geq 3$.
b) Can you also specify a satisfiable formula $F$, such that for all models $\mathcal{A}$ of $F$, we have $|U_{\mathcal{A}}| \leq 3$?

Exercise 2.2. [Resolution Completeness]
a) Does $F \models C$ imply $F \vdash_{\text{Res}} C$? Proof or counterexample!
b) Can you prove $F \models C$ by resolution?

Exercise 2.3. [Resolution of Horn-Clauses]
Can the resolvent of two Horn-clauses be a non-Horn clause?

Exercise 2.4. [Optimizing Resolution]
We call a clause $C$ trivially true if $A_i \in C$ and $\neg A_i \in C$ for some atom $A_i$. Show that the resolution algorithm remains complete if it does not consider trivially true clauses for resolution.

Exercise 2.5. [Finite Axiomatization]
Let $M_0$ and $M$ be sets of formulas. $M_0$ is called axiom schema for $M$, iff for all assignments $\mathcal{A}$: $\mathcal{A} \models M_0$ iff $\mathcal{A} \models M$.

A set $M$ is called finitely axiomatized iff there is a finite axiom scheme for $M$.

a) Are all sets of formulas finitely axiomatized? Proof or counterexample? b) Let $M = (F_i)_{i \in \mathbb{N}}$ be a set of formulas, such that for all $i$: $F_{i+1} \models F_i$, and not $F_{i+1} \models F_i$. Is $M$ finitely axiomatized?
Homework 2.1. [Definitional CNF] (3 points)
Calculate the definitional CNF of the following formula:

\[(A_1 \lor (A_2 \land \neg A_3)) \lor A_4\]

Solution:

\[(A_1 \lor (A_2 \land \neg A_3)) \lor A_4\]
\[\sim\]
\[((A_1 \lor (A_2 \land A_5)) \lor A_4) \land (A_5 \leftrightarrow \neg A_3)\]
\[\sim\]
\[((A_1 \lor A_6) \lor A_4) \land (A_5 \leftrightarrow \neg A_3) \land (A_6 \leftrightarrow (A_2 \land A_5))\]
\[\sim\]
\[(A_1 \lor A_6 \lor A_4) \land \text{CNF}(A_5 \leftrightarrow \neg A_3) \land \text{CNF}(A_6 \leftrightarrow (A_2 \land A_5))\]
\[\sim (\ast)\]
\[(A_1 \lor A_6 \lor A_4) \land (A_5 \lor A_3) \land (\neg A_5 \lor \neg A_3) \land (A_6 \lor \neg A_2 \lor \neg A_5) \land (\neg A_6 \lor A_2) \land (\neg A_6 \lor A_5)\]

\((\ast)\): By e.g. the truth table approach we get that the CNF of a formula \(L_i \leftrightarrow (L_j \land L_k)\) is \((L_i \lor \overline{L_j} \lor \overline{L_k}) \land (\overline{L_i} \lor L_j) \land (\overline{L_i} \lor L_k)\) (where \(L_i, L_j\) and \(L_k\) are literals).
Homework 2.2.  [Definitional DNF]  (5 points)

We call formulas $F$ and $F'$ equivalent if

$$\models F \iff \models F'$$

First show that

$$F[G/A] \text{ and } (A \leftrightarrow G) \rightarrow F \text{ are equivalent}$$

for any formulas $F$ and $G$ and any atom $A$, provided that $A$ does not occur in $G$. Now argue that for every formula $F$ of size $n$ there is an equivalent DNF formula $G$ of size $O(n)$.

Solution:  Suppose $A \models F[G/A]$ for any $A$. We have to consider two cases:

1. $A(A) = A(G)$. Then $A(F[G/A]) = A(F) = 1$ with the same argument as in the lecture (correctness proof for definitional CNF).

2. $A(A) \neq A(G)$. Then $A \not\models A \leftrightarrow G$.

In either case we immediately get $A \models (A \leftrightarrow G) \rightarrow F$, which completes the ‘only if’-direction.

For the other direction, assume $\models (A \leftrightarrow G) \rightarrow F$ and let $A$ be an suitable assignment for $F[G/A]$. We obtain an assingment $A'$, which is not defined for $A$ (it may be the case that $A = A'$). By the coincidence lemma, we know $A \models F[G/A]$ if $A' \models F[G/A]$. We now extend $A'$ to some assignment $A'' = A' \cup \{A \rightarrow A'(G)\}$, where $A' \models F[G/A]$ if $A'' \models F[G/A]$ by the coincidence lemma. We get $A'' \models (A \leftrightarrow G) \rightarrow F$ by assumption and thus $A'' \models F$ from the construction of $A''$. By the substitution lemma, $A''(F[G/A]) = (A''[A'(G)/A])(F) = A''(F)$. Together we have $A'' \models F[G/A]$ and thus $A \models F[G/A]$.

Construction of the definitional DNF now proceeds analogously to the definitional CNF, with the exception that new definitions of the form $A \leftrightarrow G$ are conjoined to the formula via implication and not conjunction. The last step converts the whole formula to DNF. Because $(A \leftrightarrow G) \rightarrow F \equiv (A \land \neg G) \lor (\neg A \land G) \lor F$ the size of the obtained formula is linear in the size of the original formula.

Homework 2.3.  [Compactness Theorem]  (5 points)

Suppose every subset of $S$ is satisfiable. Show that then

- every subset of $S \cup \{F\}$ is satisfiable or
- every subset of $S \cup \{\neg F\}$ is satisfiable

for any formula $F$.

Solution:  Proof by contradiction. Suppose $S \cup \{F\}$ has an unsatisfiable subset $M$ and $S \cup \{\neg F\}$ has an unsatisfiable subset $L$. We can assume that $M = M' \cup \{F\}$ and $L = L' \cup \{\neg F\}$ for some $M'$, $L'$ where $M' \subseteq S$ and $L' \subseteq S$ because every subset of $S$ is satisfiable. We additionally know that $M' \cup L'$ is satisfiable by assumption. Consider the sets

$$M' \cup L' \cup \{F\} \quad \text{and} \quad M' \cup L' \cup \{\neg F\}$$

Then one of them has to be satisfiable. (Let $A$ with $A \models M' \cup L'$. Then either $A \models F$ or $A \models \neg F$. That is, $A \models F$ or $A \models \neg F$. This directly implies that either $M$ or $L$ is satisfiable, a contradiction.)
**Homework 2.4.  [Compactness and Validity]**  
(2 points)
We say that a set of formulas \( S \) is valid if every \( F \) in \( S \) is valid. Prove or disprove:

\[
S \text{ is valid iff every finite subset of } S \text{ is valid}
\]

**Solution:** Trivial by definition:

\[
\begin{align*}
S \text{ valid & iff all } F \text{ in } S \text{ are valid} \\
&\text{iff all finite subsets of } S \text{ are valid.}
\end{align*}
\]

**Homework 2.5.  [Resolution]**  
(5 points)
Use the resolution procedure to decide if the following formulas are satisfiable. Show your work (by giving the corresponding DAG or linear derivation)!

1. \( \neg A_1 \land A_2 \land (\neg A_1 \lor A_3) \land (A_1 \lor \neg A_2 \lor A_3) \)
2. \( A_2 \land (\neg A_3 \lor A_1) \land (\neg A_1 \lor A_2) \land (\neg A_1) \land (\neg A_2 \lor A_3) \)

**Solution:**

1. 0: \( \{\neg A_1\} \)
   1: \( \{A_2\} \)
   2: \( \{\neg A_1, A_3\} \)
   3: \( \{A_1, \neg A_2, A_3\} \)
   4: \( \{\neg A_2, A_3\} \)  \(0, 3\)
   5: \( \{A_1, A_3\} \)  \(1, 3\)
   6: \( \{A_3\} \)  \(0, 5\)

No more inferences are possible and thus we conclude that the formula is satisfiable.

2. 0: \( \{A_2\} \)
   1: \( \{\neg A_3, A_1\} \)
   2: \( \{\neg A_1, A_2\} \)
   3: \( \{\neg A_1\} \)
   4: \( \{\neg A_2, A_3\} \)
   5: \( \{\neg A_1\} \)  \(0, 2\)
   6: \( \{A_3\} \)  \(0, 4\)
   7: \( \{\neg A_3\} \)  \(1, 3\)
   8: \( \square \)  \(6, 7\)

The formula is unsatisfiable.