Homework 3.1. [Equivalence] (4 points)
Let $F$ and $G$ be arbitrary formulas. (In particular, they may contain free occurrences of $x$.)
Which of the following equivalences hold? Proof or counterexample!

1. $\forall x (F \land G) \equiv \forall x F \land \forall x G$
2. $\exists x (F \land G) \equiv \exists x F \land \exists x G$

Solution:  1) holds. Assume $\mathcal{A} \models \forall x (F \land G)$,
$\iff$ for all $d \in U_{\mathcal{A}}$, we have $\mathcal{A}[d/x] \models F$ and $\mathcal{A}[d/x] \models G$,
$\iff$ for all $d_1 \in U_{\mathcal{A}}$, we have $\mathcal{A}[d_1/x] \models F$ and for all $d_2 \in U_{\mathcal{A}}$, we have $\mathcal{A}[d_2/x] \models G$
$\iff$ $\mathcal{A} \models \forall x F \land \forall x G$

2) does not hold. Let $F = P(x)$ and $G = Q(x)$, $U_{\mathcal{A}} = \{0, 1\}$, $P^A = \{0\}$, and $Q^A = \{1\}$. Clearly, $\mathcal{A} \models \exists x F \land \exists x G$ but $\mathcal{A} \not\models \exists x (F \land G)$

Homework 3.2. [Preorders] (4 points)
A reflexive and transitive relation is called preorder. In predicate logic, preorders can be characterized by the formula

$$F \equiv \forall x \forall y \forall z \ (P(x, x) \land (P(x, y) \land P(y, z) \rightarrow P(x, z)))$$

Which of the following structures are models of $F$? No proofs are required for the positive case. Give counterexamples for the negative case!

1. $U^A = \mathbb{N}$ and $P^A = \{(m, n) \mid m = n\}$
2. $U^A = 2^\mathbb{N}$ and $P^A = \{(A, B) \mid A \supseteq B\}$
3. $U^A = \mathbb{Z}$ and $P^A = \{(x, y) \mid 5 > |x - y|\}$

Solution:  1,2 are obviously preorders.

3. This is not transitive, e.g., $5 > |1 - 3|$ and $5 > |3 - 6|$, but $5 \not> |1 - 6|$
Homework 3.3. [Infinite Models] (5 points)
Consider predicate logic with equality. We use infix notation for equality, and abbreviate 
\(\neg(s = t)\) by \(s \neq t\). Moreover, we call a structure finite iff its universe is finite.

1. Specify a finite model for the formula \(\forall x (c \neq f(x) \land x \neq f(x))\).

2. Specify a model for the formula \(\forall x \forall y (c \neq f(x) \land (f(x) = f(y) \implies x = y))\).

3. Show that the above formula has no finite model.

Solution:

1. \(U^A = \{0, 1, 2\} \subseteq \mathbb{N}\) and \(c^A = 0\) and \(f^A(0) = 1 \mid f^A(n + 1) = 2 - n\)

2. \(U^A = \mathbb{N}\) and \(c^A = 0\) and \(f^A(n) = n + 1\)

3. Assume a model \(\mathcal{A}\). First note that the properties transfer to the semantic level, i.e., we have for all \(x, y \in U_\mathcal{A}\):
   
   \[
   c^A \neq f^A(x) \quad (1)
   \]
   
   \[
   f^A(x) = f^A(y) \implies x = y \quad (2)
   \]

   Now, we are in a position to show that \(U_\mathcal{A}\) is infinite. For this, we define \(x_i = (f^A)^i(c^A)\), i.e. \(i\) times \(f^A\) applied to \(c^A\). Clearly, we have \(x_i \in U_\mathcal{A}\) for all \(i\). We now show that \(i < j\) implies \(x_i \neq x_j\), immediately yielding infinity of \(U_\mathcal{A}\). We do induction on \(i\). For \(0\), we have \(x_0 = c^A \neq f^A(\ldots) = x_j\) (by (1)). For \(i + 1\), the induction hypothesis gives us \(x_i \neq x_j\), which implies \(x_{i+1} \neq x_{j+1}\) (by (2)). qed.

Homework 3.4. [Normal Forms] (3 points)
Convert the following formula to Skolem form:

\[P(x) \land \forall x \ (Q(x) \land \forall x \exists y \ P(f(x, y)))\]

Show at least the main intermediate conversion stages.

Solution:

\[
P(x) \land \forall x (Q(x) \land \forall x \exists y P(f(x, y)))
\]

\[
\sim P(x) \land \forall x_1 (Q(x_1) \land \forall x_2 \exists y P(f(x_2, y))) \quad \text{rectified}
\]

\[
\sim \exists x P(x) \land \forall x_1 (Q(x_1) \land \forall x_2 \exists y P(f(x_2, y))) \quad \text{rectified and closed}
\]

\[
\sim \exists x \forall x_1 \forall x_2 \exists y (P(x) \land (Q(x_1) \land P(f(x_2, y)))) \quad \text{RPF}
\]

\[
\sim \forall x_1 \forall x_2 (P(g) \land (Q(x_1) \land P(f(x_2, h(x_1, x_2)))) \quad \text{Skolem form}
\]
Homework 3.5. [Relation to Propositional Logic] (4 points)

Suppose that formula $F$ does not contain any variables or quantifiers. Your task is to construct a propositional formula $G$ such that $F$ is valid iff $G$ is valid. Proof that your construction does indeed fulfill this property. Is it also the case that $F$ is satisfiable iff $G$ is satisfiable?

Hints: The approach should define a new atom for every atomic formula in $F$. To construct a structure for $F$ from an assignment for $G$, it may be helpful to use as your universe the set of all terms which can be constructed from function symbols in $F$. You can assume that $F$ contains at least one constant to ensure that this universe is non-empty.

Solution: $G$ is constructed from $F$ by defining a new atom $A_{P(t_1,...,t_k)}$ for every atomic formula $P(t_1,...,t_k)$ of $G$ and then recursing over the formula structure of $F$. For instance if $F = (P(c) \land \neg Q(a,b)) \lor Q(b,c)$, then $(A_{P(c)} \land \neg A_{Q(a,b)}) \lor A_{Q(b,c)}$.

We need to construct structures for $F$ from assignments for $G$ and vice versa.

(a) Let $\mathcal{A}$ be an assignment for $G$. Let $U'_{\mathcal{A}'}$ be the set of all terms which can be constructed from parts of $F$. Define $I'_{\mathcal{A}}$ such that

- $I'_{\mathcal{A}}(f(t_1,...,t_k)) = f(t_1,...,t_k)$ for any function symbol $f$ and terms $t_1,...t_k$
- $I'_{\mathcal{A}}(P(t_1,...,t_k)) = \mathcal{A}(A_{P(t_1,...,t_k)})$ for any predicate symbol $P$ and terms $t_1,...t_k$

It is easy to show that $I'_{\mathcal{A}}(P(t_1,...,t_k)) = \mathcal{A}(A_{P(t_1,...,t_k)})$ by induction over the term structure. With induction over the formula structure of $F$ it follows that $I'_{\mathcal{A}}(F) = \mathcal{A}(G)$.

(b) Let $\mathcal{A}' = (U'_{\mathcal{A}'}, I'_{\mathcal{A}'})$ be a structure of $G$. Define $\mathcal{A}(A_{P(t_1,...,t_k)}) = I'_{\mathcal{A}'}(P(t_1,...,t_k))$ for any atom of $G$. It follows via induction over the formula structure of $F$ that $\mathcal{A}(G) = I'_{\mathcal{A}'}(F)$.

Now suppose $F$ is valid. Let $\mathcal{A}$ be any assignment for $G$. By (a) we know that we can construct a structure $\mathcal{A}'$ for $F$ such that $I'_{\mathcal{A}'}(F) = \mathcal{A}(G)$. Because $F$ is valid we have $I'_{\mathcal{A}'}(F) = \mathcal{A}(G) = 1$. Thus $G$ is valid. An analogous argument using (b) shows that $F$ is valid if $G$ is valid.

Finally, the constructions of (a) and (b) can similarly easily be used to argue that $F$ is satisfiable iff $G$ is satisfiable.