Submission of Homework: Before tutorial on June 29

Exercise 11.1.  [Natural Deduction (Warmup)]
Prove by natural deduction:

1. \((F \land G) \land H \rightarrow F \land (G \land H)\)
2. \((F \lor G) \lor H \rightarrow F \lor (G \lor H)\)

Exercise 11.2.  [Natural Deduction (Advanced)]
Prove by natural deduction:

1. \(\neg(F \land G) \rightarrow (\neg F \lor \neg G)\)
2. \(((F \rightarrow G) \rightarrow F) \rightarrow F\)

Exercise 11.3.  [Alternative \(\land E\) rule]
Show how to transform a natural deduction proof that additionally uses the following rule to one that does not use the rule:

\[
\begin{array}{c}
F, G \\
\vdots \\
F \land G \\
\hline
H
\end{array}
\]
Homework 11.1.  [Natural Deduction (Warmup)]  

Show:  

\[ \vdash_N (F \to G) \to (\neg G \to \neg F) \]

Solution:

\[
\begin{array}{c}
\neg G \quad \lbrack F \rbrack^2 \quad \lbrack F \to G \rbrack^3 \\
\hline
G \
\hline
\bot \\
\bot \quad \neg I \quad (2) \\
\hline

\neg F \\
\hline

\neg G \to \neg F \\
\hline

(F \to G) \to (\neg G \to \neg F) \quad I \quad (3)
\end{array}
\]
**Homework 11.2.  [Classical Reasoning (1)] (6 points)**

We replace rule $\bot$ of the calculus of natural deduction by either one of the following rules:

- $\frac{F \lor \neg F}{\bot}$ (law of excluded middle)
- $\frac{\neg\neg F}{F}$ (double negation elimination)

Additionally, we add the rule $\frac{\bot}{F}$ ($\bot E$). Show that the calculus of natural deduction remains complete in both cases.

**Solution:** We want to show that the $\bot$ rule can be derived from either of the two other alternatives. Thus we assume that there is a proof of the form

$$
\neg F
\vdots
\bot
$$

and we need to show that we then can also prove $F$.

- $\frac{\neg F}{\bot}$ (law of excluded middle)

  $\frac{\neg F}{\bot}$ (law of excluded middle) $\frac{F}{\bot}$ $\bot E$ (3)

- $\frac{\neg F}{\bot}$ (double negation elimination)

  $\frac{\neg F}{\bot}$ (double negation elimination)
Homework 11.3.  [Classical Reasoning (2)]  (5 points)
Assume that the calculus of natural deduction is augmented with the two rules from the last exercise. Show:

- $\vdash_N (\neg G \to \neg F) \to (F \to G)$
- $\vdash_N (\neg F \to G) \to (F \lor G)$

Hint: You can also use the law of excluded middle in the following form:

\[
\begin{array}{c|c}
F & \neg F \\
\hline
& \\
G & G \\
\end{array}
\]

\[ G \]

Solution:

- 

\[
\begin{array}{c}
\vdash_N (\neg G \to \neg F) \to (F \to G) \\
\vdash_N (\neg F \to G) \to (F \lor G)
\end{array}
\]

Homework 11.4.  [Left-Sided Sequent Calculus]  (6 points)
We want to study a modified sequent calculus where the right-hand side is always empty, i.e. where sequents are of the form $\Gamma \Rightarrow$. Give a set of rules for this calculus such that your calculus fulfills the following property and sketch a proof:

\[ \Gamma \Rightarrow \Delta \text{ iff } \Gamma, \neg \Delta \Rightarrow \]

Hint: Use induction over the length of the derivations. You can skip the cases for $\lor$ and $\land$ and instead look at $\Rightarrow$. 

**Solution:** To construct a correct set of rules, one should use the intuition that $\Gamma \Rightarrow$ means that we want to show a contradiction from $\Gamma$. This directly gives us the new axioms:

$$\frac{}{F, \neg F, \Gamma \Rightarrow} \quad \frac{}{\bot, \Gamma \Rightarrow}$$

The former ‘left’ rules then still look very much the same:

$$\frac{F, G, \Gamma \Rightarrow}{F \land G, \Gamma \Rightarrow} \land L' \quad \frac{F, \Gamma \Rightarrow \quad G, \Gamma \Rightarrow}{F \lor G, \Gamma \Rightarrow} \lor L' \quad \frac{\neg F, \Gamma \Rightarrow \quad G, \Gamma \Rightarrow}{\neg F, \Gamma \Rightarrow \quad G, \Gamma \Rightarrow} \rightarrow L'$$

For ‘right’, one way is to look at negated versions of the corresponding operators:

$$\frac{\neg F, \Gamma \Rightarrow \quad \neg G, \Gamma \Rightarrow}{\neg (F \land G), \Gamma \Rightarrow} \neg \land R' \quad \frac{\neg F, \neg G, \Gamma \Rightarrow}{\neg (F \lor G), \Gamma \Rightarrow} \neg \lor R' \quad \frac{F, \neg G, \Gamma \Rightarrow}{\neg (F \rightarrow G), \Gamma \Rightarrow} \neg \rightarrow R'$$

Considering that we have defined negation as a derived operator for sequent calculus in the lecture, the set of ‘right’ rules is somewhat peculiar. An easy way to avoid trouble is to add a rule to eliminate double negations:

$$\frac{\neg \neg F, \Gamma}{F, \Gamma}$$

Note that the “reversed” rule can already be derived from $\rightarrow R'$ (if we allow $\top$ to be dropped anywhere).

We first show that if we have a derivation of length $n$ for $\Gamma \Rightarrow \Delta$ for all $\Gamma, \Delta$, then we also have a derivation for $\Gamma, \neg \Delta \Rightarrow$.

Base case $n = 0$: If the $\bot$ axiom was used, we can directly replay the proof. Otherwise, the derivation looks like (with $\Gamma = \{F\} \cup \Gamma', \Delta = \{F\} \cup \Delta'$):

$$\frac{}{F, \Gamma' \Rightarrow F, \Delta'}$$

Again, we directly get the following by the new axiom:

$$\frac{F, \Gamma', \neg F, \neg \Delta' \Rightarrow}{\Gamma, \Delta \Rightarrow}$$

And thus:

Induction step: The induction hypothesis is that for all $\Gamma$ and $\Delta$ if we have a proof of length $n$ for $\Gamma \Rightarrow \Delta$, then we have a proof for $\Gamma, \neg \Delta \Rightarrow$. We are further given a proof for $\Gamma \Rightarrow \Delta$ of length $n + 1$. We consider the last rule used and only look at the two rules for implication:

(Case $\rightarrow L$) The proof looks like:

$$\frac{\vdots \text{length } n}{\Gamma' \Rightarrow F, \Delta \quad G, \Gamma' \Rightarrow \Delta} \quad \frac{F \Rightarrow G, \Gamma' \Rightarrow \Delta}{\rightarrow L}$$
Using the induction hypothesis two times, we get:

\[
\begin{array}{c}
\Gamma', \neg F, \neg \Delta \Rightarrow G, \Gamma', \neg \Delta \Rightarrow \\
\hline
F \rightarrow G, \Gamma', \neg \Delta \Rightarrow \\
\end{array}
\rightarrow L'
\]

(Case \(\rightarrow R\)) The proof looks like:

\[
\begin{array}{c}
\text{length } n \\
\hline
\Gamma, F \Rightarrow G, \Delta' \\
\Gamma \Rightarrow F \rightarrow G, \Delta' \rightarrow R
\end{array}
\]

Using the induction hypothesis once, we get:

\[
\begin{array}{c}
\text{length } n \\
\hline
\Gamma, F, \neg G, \neg \Delta' \Rightarrow \\
\Gamma, \neg (F \rightarrow G), \neg \Delta' \Rightarrow \\
\end{array}
\rightarrow R'
\]

The other direction: Base case \(n = 0\): If the \(\bot\) axiom was used, we can directly replay the proof. Otherwise \(\{F, \neg F\} \subseteq \Gamma \cup \neg \Delta\) for some \(F\). This gives us four cases to consider. The case where \(F \in \Gamma\) and \(Fin\Delta\) where \(\neg F \in \Gamma\) and \(\neg Fin\Delta\) follow directly. For the two other cases, we can get a proof for \(\Gamma \Rightarrow \Delta\) by first using the axiom for formulas and then one of the rules for negations.

Induction step: The induction hypothesis is that for all \(\Gamma\) and \(\Delta\) if we have a proof of length \(n\) for \(\Gamma, \neg \Delta \Rightarrow\), then we have a proof for \(\Gamma \Rightarrow \Delta\). We are further given a proof for \(\Gamma, \neg \Delta \Rightarrow\) of length \(n + 1\). We consider the last rule used and only look at the two rules for implication:

(Case \(\rightarrow L'\)) The proof looks like:

\[
\begin{array}{c}
\text{length } n \\
\hline
\Gamma', \neg F, \neg \Delta \Rightarrow G, \Gamma', \neg \Delta \Rightarrow \\
\hline
F \rightarrow G, \Gamma', \neg \Delta \Rightarrow \\
\end{array}
\rightarrow L'
\]

Using the induction hypothesis two times, we get:

\[
\begin{array}{c}
\hline
\Gamma' \Rightarrow F, \Delta \Rightarrow G, \Gamma' \Rightarrow \Delta \\
\hline
F \rightarrow G, \Gamma' \Rightarrow \Delta \\
\end{array}
\rightarrow L
\]

(Case \(\rightarrow R'\)) We now have to consider two cases because \(\neg (F \rightarrow G)\) might arise from \(\Gamma\) or \(\Delta\). The proof can look like:

\[
\begin{array}{c}
\text{length } n \\
\hline
\Gamma, F, \neg G, \neg \Delta' \Rightarrow \\
\Gamma, \neg (F \rightarrow G), \neg \Delta' \Rightarrow \\
\end{array}
\rightarrow R'
\]
Using the induction hypothesis once, we get:

\[
\begin{align*}
\Gamma, F \Rightarrow G, \Delta' & \quad \Rightarrow \\
\Gamma & \Rightarrow F \Rightarrow G, \Delta' \\
\end{align*}
\]

And the proof can look like:

\[
\begin{align*}
\vdots & \\
\text{length } n & \\
\Gamma', F, \neg G, \neg \Delta & \Rightarrow \\
\Gamma', \neg(F \rightarrow G), \neg \Delta & \Rightarrow R' \\
\end{align*}
\]

Using the induction hypothesis and \(\neg L\) once, we get:

\[
\begin{align*}
\vdots & \\
\Gamma', F \Rightarrow G, \Delta & \Rightarrow \\
\Gamma' & \Rightarrow F \Rightarrow G, \Delta \\
\Gamma', \neg(F \rightarrow G) & \Rightarrow \Delta \neg L \\
\end{align*}
\]