Propositional Logic
Horn Formulas
Efficient satisfiability checks

In the following:

- A very efficient satisfiability check for the special class of Horn formulas.
- Efficient satisfiability checks for arbitrary formulas in CNF: resolution (later).
Horn formulas

Definition
A formula $F$ in CNF is a Horn formula if every disjunction in $F$ contains at most one positive literal.

A disjunction in a Horn formula can equivalently be viewed as an implication $K \rightarrow B$ where $K$ is a conjunction of atoms or $K = \top$ and $B$ is an atom or $B = \bot$:

\[
\begin{align*}
(\neg A \lor \neg B \lor C) & \equiv (A \land B \rightarrow C) \\
(\neg A \lor \neg B) & \equiv (A \land B \rightarrow \bot) \\
A & \equiv (\top \rightarrow A)
\end{align*}
\]
Satisfiability check for Horn formulas

Input: a Horn formula $F$.

Algorithm building a model (assignment) $\mathcal{M}$:

for all atoms $A_i$ in $F$ do $\mathcal{M}(A_i) := 0$;

while $F$ has a subformula $K \rightarrow B$

such that $\mathcal{M}(K) = 1$ and $\mathcal{M}(B) = 0$

do

if $B = \bot$ then return "unsatisfiable"

else $\mathcal{M}(B) := 1$

return "satisfiable"

Maximal number of iterations of the while loop:

number of implications in $F$

Each iteration requires at most $O(|F|)$ steps.

Overall complexity: $O(|F|^2)$

[Algorithm can be improved to $O(|F|)$. See Schöning.]
Theorem

The algorithm returns “satisfiable” iff $F$ is satisfiable.

Proof Observe: if the algorithm sets $M(B) = 1$, then $A(B) = 1$ for every assignment $A$ such that $A(F) = 1$. This is an invariant.

(a) If “unsatisfiable” then unsatisfiable.

We prove unsatisfiability by contradiction. Assume $A(F) = 1$ for some $A$. Let $(A_{i_1} \land \ldots \land A_{i_k} \rightarrow \bot)$ be the subformula causing “unsatisfiable”. Since $M(A_{i_1}) = \cdots = M(A_{i_k}) = 1$, $A(A_{i_1}) = \ldots = A(A_{i_k}) = 1$. Then $A(A_{i_1} \land \ldots \land A_{i_k} \rightarrow \bot) = 0$ and so $A(F) = 0$, contradiction. So $F$ has no satisfying assignments.
(b) If “satisfiable” then satisfiable. After termination with “satisfiable”, for every subformula $K \rightarrow B$ of $F$, $M(K) = 0$ or $M(B) = 1$. Therefore $M(K \rightarrow B) = 1$ and thus $M \models F$. In fact, the invariant shows that $M$ is the minimal model of $F$. 