First-Order Logic
Herbrand Theory
Herbrand universe

The Herbrand universe $T(F)$ of a closed formula $F$ in Skolem form is the set of all terms that can be constructed using the function symbols in $F$.

In the special case that $F$ contains no constants, we first pick an arbitrary constant, say $a$, and then construct the terms.

Formally, $T(F)$ is inductively defined as follows:

- All constants occurring in $F$ belong to $T(F)$; if no constant occurs in $F$, then $a \in T(F)$ where $a$ is some arbitrary constant.
- For every $n$-ary function symbol $f$ occurring in $F$, if $t_1, t_2, \ldots, t_n \in T(F)$ then $f(t_1, t_2, \ldots, t_n) \in T(F)$.

Note: All terms in $T(F)$ are variable-free by construction!

Example

$F = \forall x \forall y \, P(f(x), g(c, y))$
Herbrand structure

Let $F$ be a closed formula in Skolem form.
A structure $\mathcal{A}$ suitable for $F$ is a Herbrand structure for $F$ if it satisfies the following conditions:

- $U_{\mathcal{A}} = T(F)$, and
- for every $n$-ary function symbol $f$ occurring in $F$ and every $t_1, \ldots, t_n \in T(F)$: $f^\mathcal{A}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

Fact

If $\mathcal{A}$ is a Herbrand structure, then $\mathcal{A}(t) = t$ for all $t \in U_{\mathcal{A}}$. 
Definition

The **matrix** of a formula $F$ is the result of removing all quantifiers (all $\forall x$ and $\exists x$) from $F$. The matrix is denoted by $F^*$. 
**Theorem**

*Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model.*

**Proof** If F has a Herbrand model then it is satisfiable.

For the other direction let $A$ be an arbitrary model of $F$. We define a Herbrand structure $T$ as follows:

- **Universe** $U_T = T(F)$
- **Function symbols** $f^T(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$
- If $F$ contains no constant: $a^A = u$ for some arbitrary $u \in U_A$
- **Predicate symbols** $(t_1, \ldots, t_n) \in P^T$ iff $(A(t_1), \ldots, A(t_n)) \in P^A$

**Claim:** $T$ is also a model of $F$. 
**Claim:** $\mathcal{T}$ is also a model of $F$.

We prove a stronger assertion:

> For every closed formula $G$ in Skolem form
> such that all fun. and pred. symbols in $G^*$ occur in $F^*$:
> if $\mathcal{A} \models G$ then $\mathcal{T} \models G$

**Proof** By induction on the number $n$ of universal quantifiers of $G$.

Basis $n = 0$. Then $G$ has no quantifiers at all.
Therefore $\mathcal{A}(G) = \mathcal{T}(G)$ (why?), and we are done.
Induction step: $G = \forall x \ H$.

\[ \mathcal{A} \models G \]
\[ \Rightarrow \text{ for every } u \in U_\mathcal{A}: \mathcal{A}[u/x](H) = 1 \]
\[ \Rightarrow \text{ for every } u \in U_\mathcal{A} \text{ of the form } u = \mathcal{A}(t) \text{ where } t \in T(G): \mathcal{A}[u/x](H) = 1 \]
\[ \Rightarrow \text{ for every } t \in T(G): \mathcal{A}[\mathcal{A}(t)/x](H) = 1 \] (substitution lemma)
\[ \Rightarrow \text{ for every } t \in T(G): \mathcal{T}(H[t/x]) = 1 \] (induction hypothesis)
\[ \Rightarrow \text{ for every } t \in T(G): \mathcal{T}[\mathcal{T}(t)/x](H) = 1 \] (substitution lemma)
\[ \Rightarrow \text{ for every } t \in T(G): \mathcal{T}[t/x](H) = 1 \] (\(\mathcal{T}\) is Herbrand structure)
\[ \Rightarrow \mathcal{T}(\forall x \ H) = 1 \] (\(U_\mathcal{T} = T(G)\))
\[ \Rightarrow \mathcal{T} \models G \]
Theorem

Let $F$ be a closed formula in Skolem form.
Then $F$ is satisfiable iff it has a Herbrand model.

What goes wrong if $F$ is not closed or not in Skolem form?
Herbrand expansion

Let $F = \forall y_1 \ldots \forall y_n F^*$ be a closed formula in Skolem form. The Herbrand expansion of $F$ is the set of formulas

$$E(F) = \{ F^*[t_1/y_1] \ldots [t_n/y_n] \mid t_1, \ldots, t_n \in T(F) \}$$

Informally: the formulas of $E(F)$ are the result of substituting terms from $T(F)$ for the variables of $F^*$ in every possible way.

Example

$E(\forall x \forall y \ P(f(x), g(c, y))) = $

Note The Herbrand expansion can be viewed as a set of propositional formulas.
Gödel-Herbrand-Skolem Theorem

Theorem

Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff its Herbrand expansion $E(F)$ is satisfiable (in the sense of propositional logic).

Proof

By the fundamental theorem, it suffices to show: $F$ has a Herbrand model iff $E(F)$ is satisfiable.

Let $F = \forall y_1 \ldots \forall y_n F^*$.

$\mathcal{A}$ is a Herbrand model of $F$

iff for all $t_1, \ldots, t_n \in T(F)$, $\mathcal{A}[t_1/y_1] \ldots [t_n/y_n](F^*) = 1$

iff for all $t_1, \ldots, t_n \in T(F)$, $\mathcal{A}(F^*[t_1/y_1] \ldots [t_n/y_n]) = 1$

iff for all $G \in E(F)$, $\mathcal{A}(G) = 1$

iff $\mathcal{A}$ is a model of $E(F)$
Herbrand’s Theorem

Theorem

Let $F$ be a closed formula in Skolem form.

$F$ is unsatisfiable iff some finite subset of $E(F)$ is unsatisfiable.

Proof

Follows immediately from the Gödel-Herbrand-Skolem Theorem and the Compactness Theorem.
Gilmore’s Algorithm

Let $F$ be a closed formula in Skolem form and let $F_1, F_2, F_3, \ldots$ be an computable enumeration of $E(F)$.

Input: $F$
$n := 0$
repeat $n := n + 1$
until $(F_1 \land F_2 \land \ldots \land F_n)$ is unsatisfiable
return “unsatisfiable”

The algorithm terminates iff $F$ is unsatisfiable.
Semi-decidiability Theorems

Theorem

(a) The unsatisfiability problem of predicate logic is (only) semi-decidable.

(b) The validity problem of predicate logic is (only) semi-decidable.

Proof
(a) Gilmore’s algorithm is a semi-decision procedure.
(The problem is undecidable. Proof later)
(b) $F$ valid iff $\neg F$ unsatisfiable.
Löwenheim-Skolem Theorem

Theorem

Every satisfiable formula of first-order predicate logic has a model with a countable universe.

Proof Let $F$ be a formula, and let $G$ be an equisatisfiable formula in Skolem form (as produced by the Normal Form transformations). Then for every set $S$:

$F$ has a model with universe $S$ iff $G$ has a model with universe $S$.

$F$ satisfiable $\Rightarrow$ $G$ satisfiable

$\Rightarrow$ $G$ has a Herbrand model $(T, l_1)$

$\Rightarrow$ $F$ has a model $(T, l_2)$

$\Rightarrow$ $F$ has a countable model

(Herbrand universes are countable)
Löwenheim-Skolem Theorem

Formulas of first-order logic cannot enforce uncountable models

Formulas of first-order logic cannot axiomatize the real numbers
because there will always be countable models