Propositional Logic

Normal Forms
Abbreviations

Until further notice:

\[ F_1 \rightarrow F_2 \] abbreviates \( \neg F_1 \lor F_2 \)

\[ \top \] abbreviates \( A_1 \lor \neg A_1 \)

\[ \bot \] abbreviates \( A_1 \land \neg A_1 \)
Definition
A literal is an atom or the negation of an atom. In the former case the literal is positive, in the latter case it is negative.
Negation Normal Form (NNF)

Definition
A formula is in negation formal form (NNF) if negation (¬) occurs only directly in front of atoms.

Example
In NNF: \( \neg A \land \neg B \)
Not in NNF: \( \neg (A \lor B) \)
Transformation into NNF

Any formula can be transformed into an equivalent formula in NNF by pushing ¬ inwards. Apply the following equivalences from left to right as long as possible:

\[-\neg F \equiv F\]
\[-(F \land G) \equiv (\neg F \lor \neg G)\]
\[-(F \lor G) \equiv (\neg F \land \neg G)\]

Example

\[-(A \land \neg B) \land C \equiv (\neg A \lor \neg \neg B) \land C \equiv ((\neg A \lor B) \land C)\]

Warning: “\(F \equiv G \equiv H\)” is merely an abbreviation for “\(F \equiv G\) and \(G \equiv H\)”

Does this process always terminate? Is the result unique?
CNF and DNF

Definition
A formula $F$ is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals:

$$F = \left( \bigwedge_{i=1}^{n} \left( \bigvee_{j=1}^{m_i} L_{i,j} \right) \right),$$

where $L_{i,j} \in \{A_1, A_2, \ldots \} \cup \{\neg A_1, \neg A_2, \ldots \}$

Definition
A formula $F$ is in disjunctive normal form (DNF) if it is a disjunction of conjunctions of literals:

$$F = \left( \bigvee_{i=1}^{n} \left( \bigwedge_{j=1}^{m_i} L_{i,j} \right) \right),$$

where $L_{i,j} \in \{A_1, A_2, \ldots \} \cup \{\neg A_1, \neg A_2, \ldots \}$
Transformation into CNF and DNF

Any formula can be transformed into an equivalent formula in CNF or DNF in two steps:

1. Transform the initial formula into its NNF
2. Transform the NNF into CNF or DNF:
   - Transformation into CNF. Apply the following equivalences from left to right as long as possible:
     \[
     (F \lor (G \land H)) \equiv ((F \lor G) \land (F \lor H)) \\
     ((F \land G) \lor H) \equiv ((F \lor H) \land (G \lor H))
     \]
   - Transformation into DNF. Apply the following equivalences from left to right as long as possible:
     \[
     (F \land (G \lor H)) \equiv ((F \land G) \lor (F \land H)) \\
     ((F \lor G) \land H) \equiv ((F \land H) \lor (G \land H))
     \]
Termination

Why does the transformation into NNF and CNF terminate?

Challenge Question: Find a weight function \( w : \text{formula} \rightarrow \mathbb{N} \) such that \( w(l.h.s.) > w(r.h.s.) \) for the equivalences

\[
\begin{align*}
\neg\neg F & \equiv F \\
\neg(F \land G) & \equiv (\neg F \lor \neg G) \\
\neg(F \lor G) & \equiv (\neg F \land \neg G) \\
(F \lor (G \land H)) & \equiv ((F \lor G) \land (F \lor H)) \\
((F \land G) \lor H) & \equiv ((F \lor H) \land (G \lor H))
\end{align*}
\]

Define \( w \) recursively:

\[
\begin{align*}
w(A_i) &= \ldots \\
w(\neg F) &= \ldots w(F) \ldots \\
w(F \land G) &= \ldots w(F) \ldots w(G) \ldots \\
w(F \lor G) &= \ldots w(F) \ldots w(G) \ldots 
\end{align*}
\]
Complexity considerations

The CNF and DNF of a formula of size $n$ can have size $2^n$

Can we do better? Yes, if we do not insist on $\equiv$.

**Definition**
Two formulas $F$ and $G$ are **equisatisfiable** if $F$ is satisfiable iff $G$ is satisfiable.

**Theorem**
*For every formula $F$ of size $n$ there is an equisatisfiable CNF formula $G$ of size $O(n)$.***