**Exercise 5.1.**  [*From Natural Deduction to Hilbert Calculus*]

Prove the following formula to a linear proof in Hilbert calculus:

\[(F \land G) \rightarrow (G \land F)\]

*Hint:* Use the deduction theorem.

**Exercise 5.2.**  [*Equivalence*]

Let \(F\) and \(G\) be arbitrary formulas. (In particular, they may contain free occurrences of \(x\).) Which of the following equivalences hold? Proof or counterexample!

1. \(\forall x (F \land G) \equiv \forall x F \land \forall x G\)
2. \(\exists x (F \land G) \equiv \exists x F \land \exists x G\)

**Exercise 5.3.**  [*Infinite Models*]

Consider predicate logic with equality. We use infix notation for equality, and abbreviate \(\neg (s = t)\) by \(s \neq t\). Moreover, we call a structure finite iff its universe is finite.

1. Specify a finite model for the formula \(\forall x (c \neq f(x) \land x \neq f(x))\).
2. Specify a model for the formula \(\forall x \forall y (c \neq f(x) \land (f(x) = f(y) \rightarrow x = y))\).
3. Show that the above formula has no finite model.
Homework 5.1.  [A Smaller Set of Axioms for Hilbert Calculus]  (6 points)

In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient:

A1 $F \rightarrow (G \rightarrow F)$
A2 $(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$
A10 $(\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)$

Derive the following axioms from the axioms above with help of $\rightarrow E$:

1. $\neg (F \rightarrow F) \rightarrow G$
2. $\neg \neg F \rightarrow F$

Homework 5.2.  [Predicate Logic]  (6 points)

1. Specify a satisfiable formula $F$ such that for all models $A$ of $F$, we have $|U_A| \geq 3$.

2. Can you also specify a satisfiable formula $F$ such that for all models $A$ of $F$, we have $|U_A| \leq 3$?

Homework 5.3.  [Orders]  (8 points)

A reflexive and transitive relation is called preorder. In predicate logic, preorders can be characterized by the formula

$$F \equiv \forall x \forall y \forall z \ (P(x, x) \land (P(x, y) \land P(y, z) \rightarrow P(x, z)))$$

Which of the following structures are models of $F$? Give an informal proof in the positive case and a counterexample for the negative case!

1. $U^A = \mathbb{N}$ and $P^A = \{(m, n) \mid m < n\}$
2. $U^A = \mathbb{Z} \times \mathbb{Z}$ and $P^A = \{((x, y), (a, b)) \mid a - b \leq x - y \}$
3. $U^A = \mathbb{R}$ and $P^A = \{(m, n) \mid m = n\}$
4. Let $Q(x, y)$ be specified as follows: $\forall x \forall y (P(x, y) \iff Q(y, x))$. Is $Q$ a preorder?

Specify the notion of partial orders, that is, preorders that additionally satisfy antisymmetry.