Exercise 1.1. [Short Questions]
Let $M$ be a set of formulas, and let $F$ and $G$ be formulas. Which of the following assertions hold?

1. If $F$ satisfiable then $M \models F$
2. $F$ is valid iff $\top \models F$
3. If $\models F$ then $M \models F$
4. If $M \models F$ then $M \cup \{G\} \models F$
5. $M \models F$ and $M \models \neg F$ cannot hold simultaneously
6. If $M \models G \rightarrow F$ and $M \models G$ then $M \models F$

Solution:
The assertions 2, 3, 4, and 6 hold.

Counterexample for 1: $F = A_1, M = \{A_2\}$

Counterexample for 5: $M = \{\bot\}$ (ex falso quodlibet)
Exercise 1.2.  [Coincidence Lemma]
Assume that for all atomic formulas $A_i$ in $F$, $\mathcal{A}(A_i) = \mathcal{A}'(A_i)$. Show that

$$\mathcal{A} \models F \iff \mathcal{A}' \models F$$

Solution:
Proof by induction over the structure of $F$. Let $\text{atoms}(F)$ denote the set of all atomic formulas $A_i$ in a formula $F$.

- Base case $F = A_i$ for some $i$:
  Observation: $\text{atoms}(A_i) = \{A_i\}$, hence $\mathcal{A}(A_i) = \mathcal{A}'(A_i)$
  $\mathcal{A} \models A_i$ iff $\mathcal{A}(A_i) = 1$ iff $\mathcal{A}'(A_i) = 1$ iff $\mathcal{A}' \models A_i$

- Base case $F = \top$: trivial

- Base case $F = \bot$: trivial

- Case $F = \neg G$ for some $G$:
  Observation: $\text{atoms}(\neg G) = \text{atoms}(G)$
  IH: $\mathcal{A} \models G$ iff $\mathcal{A}' \models G$
  $\mathcal{A} \models \neg G$ iff $\mathcal{A} \models \neg G$ iff $\mathcal{A}' \models \neg G$

- Case $F = G \lor H$ for some $G, H$:
  Observation: $\text{atoms}(F) = \text{atoms}(G) \cup \text{atoms}(H)$
  Hence, $\mathcal{A}$ and $\mathcal{A}'$ coincide on $G$ and $H$ too.
  IH 1: $\mathcal{A} \models G$ iff $\mathcal{A}' \models G$
  IH 2: $\mathcal{A} \models H$ iff $\mathcal{A}' \models H$
  Remaining proof trivial.

- Case $F = G \land H$ and remaining cases: similar

Exercise 1.3.  [Semantic Proof]
Let $\models F \rightarrow G$ where $F$ and $G$ do not share any atoms. Show that then $F$ is unsatisfiable or $G$ is a tautology (or both). Hint: you may want to use the previous result.

Solution:
Proof by contradiction. Assume that $F$ is satisfiable and $G$ is not a tautology. Obtain assignments $\mathcal{A}_F$ and $\mathcal{A}_G$ such that $\mathcal{A}_F \models F$ and $\mathcal{A}_G \not\models G$. Construct a new assignment $\mathcal{A}$ as follows:

$$\mathcal{A}(A_i) = \begin{cases} 
\mathcal{A}_F(A_i) & \text{if } A_i \in \text{atoms}(F) \\
\mathcal{A}_G(A_i) & \text{if } A_i \in \text{atoms}(G) \\
0 & \text{otherwise}
\end{cases}$$

This is well-defined, because $\text{atoms}(F) \cap \text{atoms}(G) = \emptyset$. $\mathcal{A}$ coincides with $\mathcal{A}_F$ on $F$ and with $\mathcal{A}_G$ on $G$. By coincidence lemma, $\mathcal{A} \models F$ and $\mathcal{A} \not\models G$. But $\mathcal{A} \not\models (F \rightarrow G)$, which is a contradiction to $\models F \rightarrow G$. 
Exercise 1.4. [Satisfiability Algorithms]
Check the following formulas for satisfiability using one of the algorithms seen in the lecture:

- \( F_1 = (\neg A \lor \neg D \lor B) \land D \land \neg B \land E \land (\neg D \lor \neg E \lor C) \)
- \( F_2 = (A \rightarrow C) \land (C \rightarrow E) \land (\top \rightarrow B) \land (C \land A \rightarrow D) \land (B \rightarrow A) \land \\
  (B \rightarrow E) \land (D \land E \rightarrow \bot) \)
- \( F_3 = (A \rightarrow E) \land (B \rightarrow \bot) \land (C \rightarrow B) \land (\top \rightarrow A) \land (E \land B \rightarrow C) \land (C \rightarrow D) \)

Solution:

- Rewrite to \((A \land D \rightarrow B) \land (\top \rightarrow D) \land (B \rightarrow \bot) \land (\top \rightarrow E) \land (D \land E \rightarrow C)\). Mark \(D, E, C\). Satisfiable.
- Mark \(B, A, E, C, D\). Unsatisfiable.
- Mark \(A, E\). Satisfiable.
Homework 1.1.  [CNF and DNF]  (6 points)
Use the rewriting-based procedure from the lecture to convert the following formulas $F$ and $G$ first to NNF, and then to CNF and DNF. Document each rewriting step.

$$F = \neg
\neg(\neg A_1 \land \neg \neg (A_2 \lor A_3)) \quad G = (A_1 \lor A_2 \lor A_3) \land (\neg A_1 \lor \neg A_2)$$

Homework 1.2.  [Basic equivalences]  (8 points)
Let $F$ and $G$ be formulas. Are the following statements equivalent? Proof or counterexample!

1. $\models F \leftrightarrow G$
2. $F \equiv G$

How about these two statements?

1. $F$ is valid
2. $F \equiv \top$

Homework 1.3.  [Efficient CNF satisfiability check]  (6 points)
In general, solving satisfiability for CNF formula is a hard problem. Consider the special case where clauses may only contain up to two literals. Give an efficient algorithm to check satisfiability.