Exercise 6.1. [Unused Bound Variables]

For this exercise, we first define an alternative way to evaluate a formula in a structure, based on arithmetic and set operations.

\[
\begin{align*}
\mathcal{A}(\neg F) &= 1 - \mathcal{A}(F) \\
\mathcal{A}(F \lor G) &= \max(\mathcal{A}(F), \mathcal{A}(G)) \\
\mathcal{A}(F \land G) &= \min(\mathcal{A}(F), \mathcal{A}(G)) \\
\mathcal{A}(\exists x \ F) &= \max\{\mathcal{A}[d/x](F) \mid d \in U_\mathcal{A}\} \\
\mathcal{A}(\forall x \ F) &= \min\{\mathcal{A}[d/x](F) \mid d \in U_\mathcal{A}\}
\end{align*}
\]

The evaluation for predicates and terms remain unchanged.

Equipped with this definition, prove the equivalence \( \exists x \ F \equiv F \) where \( x \) does not occur in \( F \).

**Hint:** Adapt the coincidence lemma for propositional logic (exercise 1.1) to predicate logic.

**Solution:**

*Reminder:* The coincidence lemma for propositional logic states that, assuming that for all atomic formulas \( A_i \) in \( F \), \( \mathcal{A}(A_i) = \mathcal{A}'(A_i) \), the statements \( \mathcal{A} \models F \) and \( \mathcal{A}' \models F \) are equivalent.

We formulate the coincidence lemma for predicate logic. Let \( FV(t) \) denote the variables occurring in a term and \( FV(f) \) the free variables occurring in a formula.

Let \( t \) be a term. Assume that two structures \( \mathcal{A} \) and \( \mathcal{A}' \) have identical interpretations for function and predicate symbols. Additionally, assume that for all variables \( x \in FV(t) \), \( x^\mathcal{A} = x^{\mathcal{A}'} \). Then, \( \mathcal{A}(t) = \mathcal{A}'(t) \).

**Proof:** Simple, via induction over \( t \).

Similarly, this can be applied to a formula \( F \), i.e., \( \mathcal{A}(F) = \mathcal{A}'(F) \) if \( \mathcal{A} \) and \( \mathcal{A}' \) coincide on \( FV(F) \).

Finally, we can prove the equivalence stated in the exercise. Let \( \mathcal{A} \) be an arbitrary structure suitable for \( F \). We know that \( FV(\exists x F) = FV(F) \), because \( x \) does not occur in \( F \). Hence, \( \mathcal{A} \) is also suitable for \( \exists x F \).

\[
\begin{align*}
\mathcal{A} \models \exists x F & \iff \max\{\mathcal{A}[d/x](F) \mid d \in U_\mathcal{A}\} = 1 \\
& \iff \max\{\mathcal{A}(F) \mid d \in U_\mathcal{A}\} = 1 \quad \text{(by coincidence: } x \not\in FV(F)) \\
& \iff \max\{\mathcal{A}(F)\} = 1 \\
& \iff \mathcal{A} \models F
\end{align*}
\]
Exercise 6.2. [Substitution Lemma]
Consider the following statement: “If \( F \equiv F' \), then \( F[t/x] \equiv F'[t/x] \).” Proof or counterexample.

Solution:
We can easily construct a counterexample because our definition of substitution does not take bound variables into account properly. Hence, we can “capture” bound variables during a substitution.

Counterexample:
\[
F = \forall z P(x) \\
F' = \forall y P(x) \\
F[y/x] = \forall z P(y) \\
F'[y/x] = \forall y P(y)
\]

Clearly, \( \forall x P(x) \not\equiv \forall y P(x) \). To fix this, we can forbid substitution of terms with free variables or rename bound variables during substitution.

Exercise 6.3. [Skolem Form]
Convert the following formula into – in order – a rectified formula, RPF and Skolem form.

\[
P(x) \land \forall x (Q(x) \land \forall x \exists y P(f(x,y)))
\]

Solution:

\[
\begin{align*}
P(x) & \land \forall x (Q(x) \land \exists y P(f(x,y))) \\
\sim P(x) & \land \forall x_1 Q(x_1) \land \forall x_2 \exists y P(f(x_2,y))) & \text{rectified} \\
\sim \forall x P(x) & \land \forall x_1 Q(x_1) \land \forall x_2 \exists y P(f(x_2,y))) & \text{rectified and closed} \\
\sim \exists x \forall x_1 \forall x_2 y (P(x) \land (Q(x_1) \land P(f(x_2,y)))) & \text{RPF} \\
\sim \forall x_1 \forall x_2 (P(g) \land (Q(x_1) \land P(f(x_2,h(x_1,x_2))))) & \text{Skolem form}
\end{align*}
\]

Exercise 6.4. [Herbrand Models]
Given the formula

\[
F = \forall x \forall y (P(f(x),g(y)) \land \neg P(g(x),f(y)))
\]

1. Specify a Herbrand model for \( F \).
2. Specify a Herbrand structure suitable for \( F \), which is not a model of \( F \).

Solution:
We define \( U_A = D(F) \), i.e., the Herbrand universe for \( F \). Note that we have a constant \( a \in D(F) \). We define \( f^A \) and \( g^A \) to be the Herbrand-interpretations.

1. We define \( P^A = \{(f(t_1),g(t_2)) \mid t_1, t_2 \in D(F)\} \).
2. We define \( P^A = \{(g(t_1),f(t_2)) \mid t_1, t_2 \in D(F)\} \).
Homework 6.1. [Skolem Form] (6 points)
Convert the following formulas into – in order – a rectified formula, RPF and Skolem form.

1. $\forall x \exists y \forall z \exists w (\neg P(a, w) \lor Q(f(x), y))$
2. $\forall z \exists y (P(x, g(y), z) \lor \neg \forall x Q(x))$

Homework 6.2. [Invalid Herbrand Models] (8 points)
Recall the fundamental theorem from the lecture: “Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff it has a Herbrand model”.

Explain “what goes wrong” if the precondition is violated: when $F$ is not closed or not in Skolem form. Describe both cases.

Homework 6.3. [Proof of the Fundamental Theorem] (6 points)
Recall the proof of the fundamental theorem from the lecture. Give the omitted proof for the base case (slide 6, $\mathcal{A}(G) = \mathcal{T}(G)$).