First-Order Logic

Equality
Predicate logic with equality

Predicate logic
+ distinguished predicate symbol “=” of arity 2

Semantics: A structure $\mathcal{A}$ of predicate logic with equality always maps the predicate symbol $=$ to the identity relation:

$$\mathcal{A}(=) = \{(d, d) \mid d \in U_\mathcal{A}\}$$
Expressivity

Fact

A structure is model of $\exists x \forall y \ x = y$ iff its universe is a singleton.

Theorem

Every satisfiable formula of predicate logic has a countably infinite model.

Proof Let $F$ be satisfiable.
We assume w.l.o.g. that $F = \forall x_1 \ldots \forall x_n F^*$ and the variables occurring in $F^*$ are exactly $x_1, \ldots, x_n$.
(If necessary bring $F$ into closed Skolem form).
We consider two cases:

$n = 0$. Exercise.

$n > 0$. Let $G = \forall x_1 \ldots \forall x_n F^*[f(x_1)/x_1]$, where $f$ is a function symbol that does not occur in $F^*$. $G$ is satisfiable (why?) and $T(G)$ is countably infinite. It follows from the fundamental theorem that $G$ has a countably infinite model.
Modelling equality

Let $F$ be a formula of predicate logic with equality. Let $Eq$ be a predicate symbol that does not occur in $F$. Let $E_F$ be the conjunction of the following formulas:

\[
\forall x \ Eq(x, x) \\
\forall x \forall y \ (Eq(x, y) \rightarrow Eq(y, x)) \\
\forall x \forall y \forall z \ ((Eq(x, y) \land Eq(y, z)) \rightarrow Eq(x, z))
\]

For every function symbol $f$ in $F$ of arity $n$ and every $1 \leq i \leq n$:

\[
\forall x_1 \ldots \forall x_n \forall y \ (Eq(x_i, y) \rightarrow Eq(f(x_1, \ldots, x_i, \ldots x_n), f(x_1, \ldots, y, \ldots, x_n)))
\]

For every predicate symbol $P$ in $F$ of arity $n$ and every $1 \leq i \leq n$:

\[
\forall x_1 \ldots \forall x_n \forall y (Eq(x_i, y) \rightarrow (P(x_1, \ldots, x_i, \ldots, x_n) \leftrightarrow P(x_1, \ldots, y, \ldots, x_n)))
\]

$E_F$ expresses that $Eq$ is a congruence relation on the symbols in $F$. 
Quotient structure

Definition
Let \( \mathcal{A} \) be a structure and \( \sim \) an equivalence relation on \( U_\mathcal{A} \) that is a congruence relation for all the predicate and function symbols defined by \( I_\mathcal{A} \). The quotient structure \( \mathcal{A}/\sim \) is defined as follows:

- \( U_{\mathcal{A}/\sim} = \{[u]_\sim \mid u \in U_\mathcal{A}\} \) where \( [u]_\sim = \{v \in U_\mathcal{A} \mid u \sim v\} \)
- For every function symbol \( f \) defined by \( I_\mathcal{A} \):
  \( f^{\mathcal{A}/\sim}([d_1]_\sim, \ldots, [d_n]_\sim) = [f^{\mathcal{A}}(d_1, \ldots, d_n)]_\sim \)
- For every predicate symbol \( P \) defined by \( I_\mathcal{A} \):
  \( P^{\mathcal{A}/\sim}([d_1]_\sim, \ldots, [d_n]_\sim) = P^{\mathcal{A}}(d_1, \ldots, d_n) \)
- For every variable \( x \) defined by \( I_\mathcal{A} \):
  \( x^{\mathcal{A}/\sim} = [x^{\mathcal{A}}]_\sim \)

Lemma
\( \mathcal{A}/\sim(t) = [\mathcal{A}(t)]_\sim \)

Lemma
\( \mathcal{A}/\sim(F) = \mathcal{A}(F) \)
Theorem

The formulas $E_F \land F[Eq/=]$ are equisatisfiable.

Proof We show that if $E_F \land F[Eq/=]$ is sat., then $F$ is satisfiable.
Assume $\mathcal{A} \models E_F \land F[Eq/=]$.
$\Rightarrow Eq^\mathcal{A}$ is an congruence relation.
Let $\mathcal{B} = \mathcal{A}/_{Eq^\mathcal{A}}$ (extended with $=$ interpreted as identity).
$\Rightarrow \mathcal{B} \models F[Eq/=]
$ By construction $Eq^\mathcal{B}$ is identity.
$\Rightarrow \mathcal{B}(F[Eq/=]) = \mathcal{B}(F)
$ $\Rightarrow \mathcal{B} \models F$