Propositional Logic
Equivalences
Definition (Equivalence)

Two formulas $F$ and $G$ are \textit{(semantically) equivalent} if $A(F) = A(G)$ for every assignment $A$.

We write $F \equiv G$ to denote that $F$ and $G$ are equivalent.
Exercise

Which of the following equivalences hold?

\[(A \land (A \lor B)) \equiv A\]
\[(A \land (B \lor C)) \equiv ((A \land B) \lor C)\]
\[(A \rightarrow (B \rightarrow C)) \equiv ((A \rightarrow B) \rightarrow C)\]
\[(A \rightarrow (B \rightarrow C)) \equiv ((A \land B) \rightarrow C)\]
Observation

The following connections hold:

\[ \models F \rightarrow G \quad \text{iff} \quad F \models G \]
\[ \models F \leftrightarrow G \quad \text{iff} \quad F \equiv G \]

NB: “iff” means “if and only if”
Reductions between problems (I)

▶ Validity to Unsatisfiability (and back):

\[ F \text{ valid} \iff \neg F \text{ unsatisfiable} \]
\[ F \text{ unsatisfiable} \iff \neg F \text{ valid} \]

▶ Validity to Consequence:

\[ F \text{ valid} \iff \top \models F \]

▶ Consequence to Validity:

\[ F \models G \iff F \rightarrow G \text{ valid} \]
Reductions between problems (II)

- **Validity to Equivalence:**
  \[ \text{if } F \text{ valid } \text{ iff } F \equiv \top \]

- **Equivalence to Validity:**
  \[ \text{if } F \equiv G \text{ iff } F \leftrightarrow G \text{ valid} \]
Properties of semantic equivalence

- Semantic equivalence is an equivalence relation between formulas.
- Semantic equivalence is closed under operators:

  If $F_1 \equiv F_2$ and $G_1 \equiv G_2$
  then $(F_1 \land G_1) \equiv (F_2 \land G_2)$,
  $(F_1 \lor G_1) \equiv (F_2 \lor G_2)$ and
  $\neg F_1 \equiv \neg F_2$

Equivalence relation + Closure under Operations = Congruence relation
Theorem

Let $F \equiv G$. Let $H$ be a formula with an occurrence of $F$ as a subformula. Then $H \equiv H'$, where $H'$ is the result of replacing an arbitrary occurrence of $F$ in $H$ by $G$.

Proof by induction on the structure of $H$. 
Theorem

\[(F \land F) \equiv F\]  
\[(F \lor F) \equiv F\]  
\[(F \land G) \equiv (G \land F)\]  
\[(F \lor G) \equiv (G \lor F)\]  
\[((F \land G) \land H) \equiv (F \land (G \land H))\]  
\[((F \lor G) \lor H) \equiv (F \lor (G \lor H))\]  
\[(F \land (F \lor G)) \equiv F\]  
\[(F \lor (F \land G)) \equiv F\]  

(Idempotence)

(Commutativity)

(Associativity)

(Absorption)
Equivalences (II)

\[(F \land (G \lor H)) \equiv ((F \land G) \lor (F \land H))\]  
\[(F \lor (G \land H)) \equiv ((F \lor G) \land (F \lor H))\]  \hspace{1cm} \text{(Distributivity)}

\[\neg\neg F \equiv F\]  \hspace{1cm} \text{(Double negation)}

\[\neg(F \land G) \equiv (\neg F \lor \neg G)\]

\[\neg(F \lor G) \equiv (\neg F \land \neg G)\]  \hspace{1cm} \text{(deMorgan’s Laws)}

\[\neg \top \equiv \bot\]

\[\neg \bot \equiv \top\]

\[(\top \lor G) \equiv \top\]

\[(\top \land G) \equiv G\]

\[(\bot \lor G) \equiv G\]

\[(\bot \land G) \equiv \bot\]
The symbols $\models$ and $\equiv$ are not operators in the language of propositional logic but part of the meta-language for talking about logic.

Examples:

$\mathcal{A} \models F$ and $F \equiv G$ are not propositional formulas.

$(\mathcal{A} \models F) \equiv G$ and $(F \equiv G) \leftrightarrow (G \equiv F)$ are nonsense.