First-Order Logic
Herbrand Theory
Herbrand universe

The Herbrand universe $T(F)$ of a closed formula $F$ in Skolem form is the set of all terms that can be constructed using the function symbols in $F$.

In the special case that $F$ contains no constants, we first pick an arbitrary constant, say $a$, and then construct the terms.

Formally, $T(F)$ is inductively defined as follows:

- All constants occurring in $F$ belong to $T(F)$; if no constant occurs in $F$, then $a \in T(F)$ where $a$ is some arbitrary constant.

- For every $n$-ary function symbol $f$ occurring in $F$, if $t_1, t_2, \ldots, t_n \in T(F)$ then $f(t_1, t_2, \ldots, t_n) \in T(F)$.

**Note:** All terms in $T(F)$ are variable-free by construction!

**Example**

$F = \forall x \forall y \ P(f(x), g(c, y))$
Herbrand structure

Let $F$ be a closed formula in Skolem form. A structure $\mathcal{A}$ suitable for $F$ is a **Herbrand structure** for $F$ if it satisfies the following conditions:

- $U_{\mathcal{A}} = T(F)$, and
- for every $n$-ary function symbol $f$ occurring in $F$ and every $t_1, \ldots, t_n \in T(F)$: $f^\mathcal{A}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

**Fact**

*If $\mathcal{A}$ is a Herbrand structure, then $\mathcal{A}(t) = t$ for all $t \in U_{\mathcal{A}}$.***

We call a Herbrand structure that is a model a **Herbrand model**.
Definition

The matrix of a formula $F$ is the result of removing all quantifiers (all $\forall x$ and $\exists x$) from $F$. The matrix is denoted by $F^*$. 
Fundamental theorem of predicate logic

Theorem

Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff it has a Herbrand model.

Proof

If $F$ has a Herbrand model then it is satisfiable.

For the other direction let $\mathcal{A}$ be an arbitrary model of $F$. We define a Herbrand structure $\mathcal{T}$ as follows:

- **Universe** $U_T = T(F)$
- **Function symbols** $f^T(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$
- If $F$ contains no constant: $a^\mathcal{A} = u$ for some arbitrary $u \in U_\mathcal{A}$
- **Predicate symbols** $(t_1, \ldots, t_n) \in P^T$ iff $(\mathcal{A}(t_1), \ldots, \mathcal{A}(t_n)) \in P_\mathcal{A}$

Claim: $\mathcal{T}$ is also a model of $F$. 
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We prove a stronger assertion:

For every closed formula $G$ in Skolem form such that all fun. and pred. symbols in $G$ occur in $F$:

if $\mathcal{A} \models G$ then $\mathcal{T} \models G$

Proof By induction on the number $n$ of universal quantifiers of $G$.

Basis $n = 0$. Then $G$ has no quantifiers at all. Therefore $\mathcal{A}(G) = \mathcal{T}(G)$ (why?), and we are done.
Induction step: $G = \forall x H$.

\[
\begin{align*}
\mathcal{A} & \models G \\
\Rightarrow & \text{ for every } u \in U_\mathcal{A}: \mathcal{A}[u/x](H) = 1 \\
\Rightarrow & \text{ for every } u \in U_\mathcal{A} \text{ of the form } u = \mathcal{A}(t)
\quad \text{where } t \in T(G): \mathcal{A}[u/x](H) = 1 \\
\Rightarrow & \text{ for every } t \in T(G): \mathcal{A}[\mathcal{A}(t)/x](H) = 1 \\
\Rightarrow & \text{ for every } t \in T(G): \mathcal{T}(\mathcal{A}(t)/x)(H) = 1 \\
\Rightarrow & \text{ for every } t \in T(G): \mathcal{T}[\mathcal{T}(t)/x](H) = 1 \\
\Rightarrow & \text{ for every } t \in T(G): \mathcal{T}[t/x](H) = 1 \\
\Rightarrow & \mathcal{T}(\forall x H) = 1 \\
\Rightarrow & \mathcal{T} \models G
\end{align*}
\]
Theorem

Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff it has a Herbrand model.

What goes wrong if $F$ is not closed or not in Skolem form?
Herbrand expansion

Let \( F = \forall y_1 \ldots \forall y_n F^* \) be a closed formula in Skolem form. The **Herbrand expansion** of \( F \) is the set of formulas

\[
E(F) = \{ F^*[t_1/y_1] \ldots [t_n/y_n] \mid t_1, \ldots, t_n \in T(F) \}
\]

Informally: the formulas of \( E(F) \) are the result of substituting terms from \( T(F) \) for the variables of \( F^* \) in every possible way.

**Example**

\[
E(\forall x \forall y \ P(f(x), g(c, y))) =
\]

**Note** The Herbrand expansion can be viewed as a set of propositional formulas.
Theorem

Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff its Herbrand expansion $E(F)$ is satisfiable (in the sense of propositional logic).

Proof By the fundamental theorem, it suffices to show: $F$ has a Herbrand model iff $E(F)$ is satisfiable.

Let $F = \forall y_1 \ldots \forall y_n F^*$.

$\mathcal{A}$ is a Herbrand model of $F$

iff for all $t_1, \ldots, t_n \in T(F)$, $\mathcal{A}[t_1/y_1] \ldots [t_n/y_n](F^*) = 1$

iff for all $t_1, \ldots, t_n \in T(F)$, $\mathcal{A}(F^*[t_1/y_1] \ldots [t_n/y_n]) = 1$

iff for all $G \in E(F)$, $\mathcal{A}(G) = 1$

iff $\mathcal{A}$ is a model of $E(F)$
Herbrand’s Theorem

Theorem
Let $F$ be a closed formula in Skolem form.
$F$ is unsatisfiable iff some finite subset of $E(F)$ is unsatisfiable.

Proof  Follows immediately from the Gödel-Herbrand-Skolem Theorem and the Compactness Theorem.
Gilmore’s Algorithm

Let $F$ be a closed formula in Skolem form and let $F_1, F_2, F_3, \ldots$ be an computable enumeration of $E(F)$.

Input: $F$

$n := 0$;

repeat $n := n + 1$;

until $(F_1 \land F_2 \land \ldots \land F_n)$ is unsatisfiable;

return “unsatisfiable”

The algorithm terminates iff $F$ is unsatisfiable.
Semi-decidiability Theorems

Theorem

(a) The unsatisfiability problem of predicate logic is (only) semi-decidable.

(b) The validity problem of predicate logic is (only) semi-decidable.

Proof

(a) Gilmore’s algorithm is a semi-decision procedure.
(The problem is undecidable. Proof later)

(b) $F$ valid iff $\neg F$ unsatisfiable.
Löwenheim-Skolem Theorem

Theorem
Every satisfiable formula of first-order predicate logic has a model with a countable universe.

Proof Let \( F \) be a formula, and let \( G \) be an equisatisfiable formula in Skolem form (as produced by the Normal Form transformations). Fact: Every model of \( G \) is a model of \( F \). (Check this!)

\[
\begin{align*}
F \text{ satisfiable} & \implies G \text{ satisfiable} \\
& \implies G \text{ has a Herbrand model } \mathcal{T} \\
& \implies F \text{ also has that model } \mathcal{T} \\
& \implies F \text{ has a countable model} \\
& \quad \text{(Herbrand universes are countable)}
\end{align*}
\]
Löwenheim-Skolem Theorem

Formulas of first-order logic cannot enforce uncountable models

Formulas of first-order logic cannot axiomatize the real numbers because there will always be countable models