First-Order Logic
Undecidability

[Cutland, *Computability*, Section 6.5.]
Aim:
Show that validity of first-order formulas is undecidability

Method:
Reduce the halting problem to validity of formulas by expressing program behaviour as formulas

Logical formulas can talk about computations!
Register machine programs (RMPs)

A register machine program is a sequence of instructions $I_1, \ldots, I_s$. The instructions manipulate registers $R_i$ ($i = 1, 2, \ldots$) that contain (unbounded!) natural numbers. There are 4 instructions:

\begin{align*}
R_n &:= 0 \\
R_n &:= R_n + 1 \\
R_n &:= R_m \\
\text{IF } R_m &= R_n \text{ GOTO } p
\end{align*}

Assumption: all jumps in a program go to 1, \ldots, $s + 1$; execution terminates when the PC is $s + 1$.

Let $r$ be the maximal index of any register used in a program $P$. Then the state of $P$ during execution can be described by a tuple of natural numbers

$$(n_1, \ldots, n_r, k)$$

where $n_i$ is the contents of $R_i$ and $k$ is the PC (the number of the next instruction to be executed).
Undecidability

Theorem (Undecidability of the halting problem for RMPs)
It is undecidable if a given register machine program terminates when started in state \((0, \ldots, 0, 1)\).

We reduce the halting problem for RMPs to the validity problem for first-order formulas.

Notation:
\(P(0) \downarrow = \text{"RMP } P \text{ started in state } (0, \ldots, 0, 1) \text{ terminates"}\)

Theorem
Given an RMP \(P\) we can effectively construct a closed formula \(\varphi_P\) such that \(P(0) \downarrow \text{ iff } \models \varphi_P\).