Submission of homework: Before tutorial on 03.07.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 12.1. [Substitution in Sequent Calculus]
Prove that $\Gamma \vdash \Delta$ implies $\Gamma[t/x] \vdash \Delta[t/x]$, where, for a set of formulas $\Gamma$, we define $\Gamma[t/x]$ to be $\{F[t/x] \mid F \in \Gamma\}$, i.e. free occurrences of $x$ are replaced by $t$. Give two different proofs:

1. A syntactic proof, transforming the proof tree of $\Gamma \vdash \Delta$.
2. A semantic proof, using correctness and completeness of $\Gamma$.

Exercise 12.2. [QE for DLO]
Use the quantifier-elimination procedure for DLOs to check whether the following formula is a member of $Th(DLO)$:

$$\exists x \forall y \exists z((x < y \lor z < x) \land y < z)$$

Exercise 12.3. [Fourier–Motzkin Elimination]

1. $\exists x \exists y(2 \cdot x + 3 \cdot y = 7 \land x < y \land 0 < x)$
2. $\exists x \exists y(3 \cdot x + 3 \cdot y < 8 \land 8 < 3 \cdot x + 2 \cdot y)$

Exercise 12.4. [Ferrante–Rackoff Elimination]

$$\exists x (\exists y(x = 2 \cdot y) \rightarrow (2 \cdot x \geq 0 \lor 3 \cdot x < 2))$$
Homework 12.1. [Subtraction Logic] (8 points)

We consider a fragment of linear arithmetic, in which atomic formulas only take the form $x - y \leq c$ for variables $x$ and $y$, and $c \in \mathbb{Q}$.

For a finite set $S$ of such difference constraints, we can define a corresponding inequality graph $G(V, E)$, where $V$ is the set of variables of $S$, and $E$ consists of all the edges $(x, y)$ with weight $c$ for all constraints $x - y \leq c$ of $S$. Show that the conjunction of all constraints from $S$ is satisfiable iff $G$ does not contain a negative cycle.

How can you use this theorem to obtain a procedure for deciding whether a formula is a member of this fragment?

Homework 12.2. [Min, Max, Abs] (6 points)

1. Show that $\text{Th}(\mathbb{R}, 0, 1, <, =, +, \text{min}, \text{max})$ is decidable, where min and max return the minimum and maximum of two values.

2. Show that $\text{Th}(\mathbb{R}, 0, 1, <, =, +, \text{min}, \text{max}, |\cdot|)$ is decidable, where $|\cdot|$ is the absolute value.

Homework 12.3. [Optimizing DLO] (6 points)

DLO suffers from a heavy performance loss because after each step, a DNF needs to be reconstructed. We want to study an optimization that may avoid this under some circumstances.

Assume that we want to eliminate an $\exists x F$ where

- $F$ is closed (except for $x$),
- $F$ contains no negations, and
- there are only lower or only upper bounds for $x$ in $F$.

Then, $\exists x F \equiv \top$. Prove correctness of this optimization.