Exercise 1.1. [Short Questions]
Let $M$ be a set of formulas, and let $F$ and $G$ be formulas. Which of the following assertions hold?

1. If $F$ satisfiable then $M \models F$
2. $F$ is valid iff $\top \models F$
3. If $\models F$ then $M \models F$
4. If $M \models F$ then $M \cup \{G\} \models F$
5. $M \models F$ and $M \models \neg F$ cannot hold simultaneously
6. If $M \models G \rightarrow F$ and $M \models G$ then $M \models F$

Solution:
The assertions 2, 3, 4, and 6 hold.

Counterexample for 1: $F = A_1, M = \{A_2\}$

Counterexample for 5: $M = \{\bot\}$ (ex falso quodlibet)
Exercise 1.2. [Coincidence Lemma]
Assume that for all atomic formulas $A_i$ in $F$, $A(A_i) = A'(A_i)$. Show that
\[ A \models F \quad \text{iff} \quad A' \models F \]

Solution:
Proof by induction over the structure of $F$. Let $\text{atoms}(F)$ denote the set of all atomic formulas $A_i$ in a formula $F$.

- Base case $F = A_i$ for some $i$:
  Observation: $\text{atoms}(A_i) = \{A_i\}$, hence $A(A_i) = A'(A_i)$
  $A \models A_i$ iff $A(A_i) = 1$ iff $A'(A_i) = 1$ iff $A' \models A_i$

- Base case $F = \top$: trivial

- Base case $F = \bot$: trivial

- Case $F = \neg G$ for some $G$:
  Observation: $\text{atoms}(\neg G) = \text{atoms}(G)$
  IH: $A \models G$ iff $A' \models G$
  $A \models \neg G$ iff $A \not\models G$ iff $A' \not\models G$ iff $A' \models \neg G$

- Case $F = G \lor H$ for some $G, H$:
  Observation: $\text{atoms}(F) = \text{atoms}(G) \cup \text{atoms}(H)$
  Hence, $A$ and $A'$ coincide on $G$ and $H$ too.
  IH 1: $A \models G$ iff $A' \models G$
  IH 2: $A \models H$ iff $A' \models H$
  Remaining proof trivial.

- Case $F = G \land H$ and remaining cases: similar

Exercise 1.3. [Semantic Proof]
Let $\models F \to G$ where $F$ and $G$ do not share any atoms. Show that then $F$ is unsatisfiable or $G$ is a tautology (or both). Hint: you may want to use the previous result.

Solution:
Proof by contradiction. Assume that $F$ is satisfiable and $G$ is not a tautology. Obtain assignments $A_F$ and $A_G$ such that $A_F \models F$ and $A_G \not\models G$. Construct a new assignment $A$ as follows:

\[
A(A_i) = \begin{cases} 
A_F(A_i) & \text{if } A_i \in \text{atoms}(F) \\
A_G(A_i) & \text{if } A_i \in \text{atoms}(G) \\
0 & \text{otherwise}
\end{cases}
\]

This is well-defined, because $\text{atoms}(F) \cap \text{atoms}(G) = \emptyset$. $A$ coincides with $A_F$ on $F$ and with $A_G$ on $G$. By coincidence lemma, $A \models F$ and $A \not\models G$. But $A \not\models (F \to G)$, which is a contradiction to $\models F \to G$. 

Homework 1.1. [CNF and DNF] (6 points)
Use the rewriting-based procedure from the lecture to convert the following formulas $F$ and $G$ first to NNF, and then to CNF and DNF. Document each rewriting step.

$$F = \neg\neg(\neg A_1 \land \neg\neg(A_2 \lor A_3)) \quad G = (A_1 \lor A_2 \lor A_3) \land (\neg A_1 \lor \neg A_2)$$

Homework 1.2. [Basic equivalences] (8 points)
Let $F$ and $G$ be formulas. Are the following statements equivalent? Proof or counterexample!

1. $\models F \leftrightarrow G$
2. $F \equiv G$

How about these two statements?

1. $F$ is valid
2. $F \equiv \top$

Homework 1.3. [Efficient CNF satisfiability check] (6 points)
In general, solving satisfiability for CNF formula is a hard problem. Consider the special case where clauses may only contain up to two literals. Give an efficient algorithm to check satisfiability.