Exercise 2.1. [Resolution Completeness]

1. Does $F \models C$ imply $F \vdash_{\text{Res}} C$? Proof or counterexample!

2. Can you prove $F \models C$ by resolution?

Solution:
Resolution can be used to prove that $F \models \bot$. From the lecture notes: $F$ unsatisfiable iff $F \vdash_{\text{Res}} \Box$.

1. Counterexample: $F = \emptyset, C = \{A, \neg A\}$

2. $F \models C$
   - if $\vdash \neg F \lor C$
   - if $\neg F \lor C$ tautology
   - if $\neg(\neg F \lor C)$ unsatisfiable
   - if $\neg(\neg F \lor C) \vdash_{\text{Res}} \Box$

Exercise 2.2. [Resolution of Horn-Clauses]
Can the resolvent of two Horn-clauses be a non-Horn clause?

Solution:
No. Proof: Let $C_1, C_2$ be two Horn clauses. Both of them have at most one positive literal. Without loss of generality, let $A_i$ be the positive literal occurring in $C_1$. Hence, $\neg A_i$ occurs in $C_2$. From the Horn clause property, we get that there is no other positive literal in $C_1$ and at most one in $C_2$. The resolvent is $C' = (C_1 - \{A_i\}) \cup (C_2 - \{\neg A_i\})$. We count the positive literals: None in $(C_1 - \{A_i\})$ and at most one in $(C_2 - \{\neg A_i\})$. Hence, at most one positive literal in $C'$. 
**Exercise 2.3.  [Optimizing Resolution]**

We call a clause $C$ *trivially true* if $A_i \in C$ and $\neg A_i \in C$ for some atom $A_i$. Show that the resolution algorithm remains complete if it does not consider trivially true clauses for resolution.

**Solution:**

Completeness: If $F$ unsatisfiable, then $F \vdash_{\text{Res'}} \Box$.

First we prove a lemma: If $F$ is unsatisfiable and contains a trivially true clause $C$, then $F' = F - C$ is still unsatisfiable. Proof by contraposition. Assume $F - C$ is satisfiable. Because $C$ is trivially satisfiable, $(F - C) \cup C = F$ is satisfiable. It follows that we can construct a $F'$ that contains no trivial clauses.

Assume that $F$ is unsatisfiable. We modify the completeness proof of resolution. Recall that that proof proceeds by induction on the number of atomic formulas in $F$. We strengthen the induction by mandating that $F$ contains no trivially true clauses. Initially, this is guaranteed by the lemma. If $F$ is an unsatisfiable set of clauses containing $n + 1$ atomic formulas, we construct $F_0$ and $F_1$ by setting $A_{n+1}$ to 0 or 1, respectively. Both $F_0$ and $F_1$ are unsatisfiable. Also, neither $F_0$ nor $F_1$ contain trivial clauses. By induction hypothesis, we can obtain resolution proofs such that $F_0 \vdash_{\text{Res'}} \Box$ and $F_1 \vdash_{\text{Res'}} \Box$. Constructing the new resolution proof for $F$ introduces no new trivial clauses.

**Exercise 2.4.  [Finite Axiomatization]**

Let $M_0$ and $M$ be sets of formulas. $M_0$ is called *axiom schema* for $M$, iff for all assignments $\mathcal{A}$: $\mathcal{A} \models M_0$ iff $\mathcal{A} \models M$.

A set $M$ is called *finitely axiomatized* iff there is a finite axiom schema for $M$.

1. Are all sets of formulas finitely axiomatized? Proof or counterexample!

2. Let $M = \langle F_i \rangle_{i \in \mathbb{N}}$ be a sequence of formulas, such that for all $i$: $F_{i+1} \models F_i$, and not $F_i \models F_{i+1}$. Is $M$ finitely axiomatized?

**Solution:**

1. Counterexample: $M = \{A_1, A_1 \land A_2, A_1 \land A_2 \land A_3, \ldots \}$. Assume there is a finite axiom schema $M_0$, $M_0$ can only contain finitely many atoms. Let $\mathcal{A}$ be an assignment that maps all $A_i$ in $M_0$ to 1, but all other $A_i$ to 0. Hence, $\mathcal{A} \models M_0$ but not $\mathcal{A} \models M$.

2. The same counterexample as above works here.
**Exercise 2.5. [Compactness Theorem]**
Suppose every finite subset of $S$ is satisfiable. Show that then

- every finite subset of $S \cup \{F\}$ is satisfiable or
- every finite subset of $S \cup \{\neg F\}$ is satisfiable

for any formula $F$.

**Solution:**
Proof by contradiction. Suppose $S \cup \{F\}$ has an unsatisfiable subset $M$ and $S \cup \{\neg F\}$ has an unsatisfiable subset $L$. We can assume that $M = M' \cup \{F\}$ and $L = L' \cup \{\neg F\}$ for some $M'$, $L'$ where $M' \subseteq S$ and $L' \subseteq S$ because every subset of $S$ is satisfiable. We additionally know that $M' \cup L'$ is satisfiable by assumption. Consider the sets

$$M' \cup L' \cup \{F\} \quad \text{and} \quad M' \cup L' \cup \{\neg F\}$$

Then one of them has to be satisfiable. (Let $\mathcal{A}$ with $\mathcal{A} \models M' \cup L'$. Then either $\mathcal{A} \models F$ or $\mathcal{A} \not\models F$. That is, $\mathcal{A} \models F$ or $\mathcal{A} \models \neg F$.) This directly implies that either $M$ or $L$ is satisfiable, a contradiction.
Homework 2.1.  [Resolution] (4 points)
Use the resolution procedure to decide if the following formulas are satisfiable. Show your
work (by giving the corresponding DAG or linear derivation)!

1. \((A_1 \lor A_2 \lor \neg A_3) \land \neg A_1 \land (A_1 \lor A_2 \lor A_3) \land (A_1 \lor \neg A_2)\)
2. \((\neg A_1 \lor A_2) \land (\neg A_2 \lor A_3) \land (A_1 \lor \neg A_3) \land (A_1 \lor A_2 \lor A_3)\)

Homework 2.2.  [Negative Resolution] (6 points)
We call a clause \(C\) negative if it only contains negative clauses. Show that resolution remains
complete if it only resolves two clauses if one of them is negative.

Homework 2.3.  [Satisfiability] (5 points)
Check the following formulas for satisfiability using one of the algorithms seen in the lecture:

1. \((A \lor \neg B \lor \neg D \lor \neg E) \land (\neg B \lor C) \land B \land (\neg C \lor D) \land (\neg D \lor E)\)
2. \(\neg((A \rightarrow B) \land (B \rightarrow A)) \rightarrow (A \leftrightarrow B)\)
3. \((A \rightarrow E) \land (B \rightarrow \bot) \land (C \rightarrow B) \land (\top \rightarrow A) \land (A \land B \rightarrow C) \land (C \rightarrow D)\)

Show your work! Remember to give a model for satisfiable formulas.
Homework 2.4. [Application of the Compactness Theorem] (5 points)

A finitely branching tree has the following structure:

- There is exactly one root node.
- Every node has a finite number of children.

We assign the root node the level 0 and the children of a node at level \( n \) the level \( n + 1 \). Let \( T_n \) denote the set of all nodes at level \( n \), and \( T \) the set of all nodes, i.e. \( T = \bigcup_{n \in \mathbb{N}} T_n \). Let \( P_t \) for \( t \in T \) be the set of parent nodes of a node, i.e. \( t \) is a child (or grand-child, ...) of all \( t' \in P_t \). A path is a sequence of connected nodes, starting from the root node.

Prove the following lemma using the compactness theorem: Every countably infinite, finitely branching tree has an infinite path.

Hint: Use the following template for the proof.

1. Fix a set of tree nodes \( T \). This set is (countably) infinite. You can assume that the sets \( T_n \) and the sets \( P_t \) are given.

2. For each node \( t \in T \), let \( A_t \) be an atomic formula. If an assignment \( \mathcal{A} \) makes \( A_t \) true, the node \( t \) is part of the path.

3. Define a set of propositions \( S \) that together guarantee the existence of an infinite path. That set is composed of three subsets:

   (a) For each level \( n \in \mathbb{N} \), a node \( t \in T_n \) is part of the path.

   (b) If a node \( t \) is part of the path, so are all of its parent nodes \( t' \in P_t \).

   (c) For each level \( n \in \mathbb{N} \), there is at most one node of level \( n \) part of the path.

4. Show that any finite subset of \( S' \subseteq S \) is satisfiable by constructing an assignment such that \( \mathcal{A}_{S'} \models S' \). Consider the largest \( n \) for which a proposition from subset (a) is contained in \( S' \).

5. Hence, \( S \) is satisfiable. Show that a model \( \mathcal{A} \models S \) represents an infinite path in \( T \).