Exercise 3.1.  [System G1c]
An alternative definition of the sequent calculus (“G1c”) is defined as follows:

**Axioms**

- **Ax**  \( A \Rightarrow A \)
- **L⊥ ⊥ ⇒**

**Rules for weakening (W) and contraction (C)**

- **LW** \( \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \)
- **RW** \( \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \)
- **LC** \( \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \)
- **RC** \( \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} \)

**Rules for the logical operators**

- **L∧** \( \frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \land A_1, \Gamma \Rightarrow \Delta} \) (\( i = 0, 1 \))
- **RA** \( \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \land B} \)
- **L∨** \( \frac{A, \Gamma \Rightarrow \Delta}{A \lor B, \Gamma \Rightarrow \Delta} \)
- **RV** \( \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A \lor A_1} \) (\( i = 0, 1 \))
- **L→** \( \frac{\Gamma \Rightarrow \Delta, A}{A \Rightarrow B, \Gamma \Rightarrow \Delta} \)
- **R→** \( \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \Rightarrow B} \)

Notably, weakening and contraction are built-in rules. Show that sequent calculus can be simulated by G1c, i.e., \( \vdash_G \Gamma \Rightarrow \Delta \) implies \( \vdash_{G1c} \Gamma \Rightarrow \Delta \).

**Solution:**
We consider two rules, \( \land L \) and \( \neg R \). We show how those can be simulated in G1c.

- **L∧** \( \frac{F, G, \Gamma \Rightarrow \Delta}{F, F \land G, \Gamma \Rightarrow \Delta} \)
- **LC** \( \frac{F \land G, F \land G, \Gamma \Rightarrow \Delta}{F \land G, \Gamma \Rightarrow \Delta} \)
- **RW** \( \frac{F, \Gamma \Rightarrow \Delta}{F, \Gamma \Rightarrow \bot, \Delta} \)
- **R→** \( \frac{\Gamma \Rightarrow F \Rightarrow \bot, \Delta}{\Gamma \Rightarrow \neg F, \Delta} \)
Exercise 3.2.  [Cut Elimination, Semantically]
Semantically prove the admissibility of the following rule:

\[
\Gamma \vdash F, \Delta \quad \text{and} \quad \Gamma \vdash G, \Gamma \vdash \Delta \quad \text{then} \quad \Gamma \vdash G, \Gamma \vdash \Delta
\]

Solution:
To prove this semantically, we have to show that given \(|\Gamma \Rightarrow F, \Delta|\) and \(|F, \Gamma \Rightarrow \Delta|, \ |\Gamma \Rightarrow \Delta|\) holds. In this case, an even stronger property holds: precedent and antecedent are equivalent. That is, \((G \rightarrow F \lor D) \land (F \land G \rightarrow D) \equiv G \rightarrow D\). We can prove this with sequent calculus:\(^1\)

\[
\begin{array}{ll}
G \vdash G, F, D & G, F \vdash F, D \\
G, D \vdash F, D & (v1)
\end{array}
\]

\[
\begin{array}{ll}
G, G \rightarrow F \lor D \vdash F, D & G, G \rightarrow F \vdash F, D \\
(->) & (-l)
\end{array}
\]

\[
\begin{array}{ll}
G, G \rightarrow F \forall D \vdash F \land G, D & G, G \rightarrow G \lor D \vdash D \\
(G \rightarrow F \forall D) \land (F \land G \rightarrow D) \vdash D & (-r)
\end{array}
\]

\[
\begin{array}{ll}
(\neg) & (\rightarrow)
\end{array}
\]

The other direction is similar:\(^2\)

\[
\begin{array}{ll}
G \vdash G, F, D & G, D \vdash F, D \\
(-l) & (-l)
\end{array}
\]

\[
\begin{array}{ll}
G, G \rightarrow D \vdash F, D & (\lor r) \\
G, G \rightarrow D \vdash F \forall D & (-r)
\end{array}
\]

\[
\begin{array}{ll}
G \rightarrow D \vdash G \forall D & G \rightarrow D \vdash F \land G \rightarrow D \\
(-r) & (Ar)
\end{array}
\]

\[
\begin{array}{ll}
G \rightarrow D \vdash (G \rightarrow F \forall D) \land (F \land G \rightarrow D) & (-r)
\end{array}
\]

\[
\begin{array}{ll}
\vdash (G \rightarrow D) \rightarrow (G \rightarrow F \forall D) \land (F \land G \rightarrow D) &
\end{array}
\]

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\(^1\)http://logitext.mit.edu/proving/+.28G+.2D.3E+F+.5C.2F+D.29+.2F.5C+.28F+.2F.5C+G+.2D.3E+D.29+.2D.3E+.28G+.E2.86.92+D.29

\(^2\)http://logitext.mit.edu/proving/+.28G+.2D.3E+D.29+.2D.3E+.28G+.2D.3E+F+.5C.2F+D.29+.2F.5C+.28F+.2F.5C+G+.2D.3E+D.29.29
Exercise 3.3. [More Connectives]
Define sequent rules for the logical connectives “nand” ($\neg \wedge$) and “xor” ($\otimes$).

Solution:
The simplest way to derive the sequent rules is to consider the definition of $\neg \wedge$ and $\otimes$.

\[
F \neg \wedge G \equiv \neg (F \wedge G)
\]
\[
F \otimes G \equiv (F \wedge \neg G) \lor (\neg F \wedge G)
\]

One can apply sequent calculus rules on the right-hand sides and simplify accordingly.

\[
\begin{align*}
\text{\texttt{\neg L}} & \quad \text{\Gamma} \Rightarrow \Delta, F & \quad \text{\Gamma} \Rightarrow \Delta, G \\
& \quad \text{\Gamma, F \neg \wedge G} \Rightarrow \Delta \\
\text{\texttt{\neg R}} & \quad \Gamma, F, G \Rightarrow \Delta \\
& \quad \Gamma \Rightarrow \Delta, F \neg \wedge G \\
\text{\texttt{\otimes L}} & \quad \Gamma, F \Rightarrow \Delta, G & \quad \Gamma, G \Rightarrow \Delta, F \\
& \quad \text{\Gamma, F \otimes G} \Rightarrow \Delta \\
\text{\texttt{\otimes R}} & \quad \Gamma \Rightarrow \Delta, F, G & \quad \Gamma, F, G \Rightarrow \Delta \\
& \quad \Gamma \Rightarrow \Delta, F \otimes G
\end{align*}
\]

Exercise 3.4. [Intermediate Formulas]
Let $F, G$ be formulas such that $F \models G$. Prove that there is an intermediate formula $H$ such that the following three conditions hold:

1. $H$ contains only atomic formulas that occur in both $F$ and $G$
2. $F \models H$
3. $H \models G$

How can $H$ be constructed?

Solution:
This theorem is called “Craig’s interpolation theorem”. We call $H$ the interpolant.

The proof proceeds by induction on the number of elements $n$ in $\text{atoms}(F) \setminus \text{atoms}(G)$.

- Base case: $n = 0$.
  Hence, $|\text{atoms}(F) \setminus \text{atoms}(G)| = 0$. Hence, $\text{atoms}(F) \subseteq \text{atoms}(F) \cap \text{atoms}(G)$. $F$ is a suitable interpolant.

- Inductive step: $n \rightarrow n + 1$.
  There is at least an atomic formula $A$ such that $A \in \text{atoms}(F)$ but $A \not\in \text{atoms}(G)$. We define a new formula $F'$ that is the disjunction of $F$ where $A$ is replaced with $\top$ and $F$ where $A$ is replaced with $\bot$:

\[
F' = F[\top/A] \lor F[\bot/A]
\]

Intuitively, $F'$ is a “case distinction” on $A$. Observe that $A \not\in \text{atoms}(F')$. Also, $|\text{atoms}(F') \setminus \text{atoms}(G)| = n$.

Use the induction hypothesis to obtain an interpolant $H$ for $F'$ and $G$ with $F' \models H$ and $H \models G$.

We need to show that $F \models H$. This is trivial because $F \models F'$. 

**Homework 3.1.**  **[Sequent Calculus]**  (2 points)
Prove the formula \(((A \rightarrow \bot) \rightarrow A) \rightarrow A \) in System G1c.

**Homework 3.2.**  **[Inversion Rules]**  (6 points)
Show that the following inversion rules are admissible:

\[
\begin{align*}
F \land G, \Gamma \Rightarrow \Delta & \quad \Gamma \Rightarrow F \rightarrow G, \Delta \\
F, G, \Gamma \Rightarrow \Delta & \quad F, \Gamma \Rightarrow G, \Delta
\end{align*}
\]

**Homework 3.3.**  **[Sequent Prover]**  (12 points)
Implement a sequent calculus prover in a high-level programming language, and test it for examples from this exercise sheet, the lecture, or your own.

*Submission:* Source code for prover and tests, README file containing instructions for how to build the prover and reproduce the tests; by email to hupel@in.tum.de. Allowed languages are: Haskell, OCaml, Java, Scala, Rust, Prolog, C++, Python. Only the standard library (i.e. no additional packages) may be used.