Exercise 6.1.  [Equivalence]
Let $F$ and $G$ be arbitrary formulas. (In particular, they may contain free occurrences of $x$.)
Which of the following equivalences hold? Proof or counterexample!

1. $\forall x(F \land G) \equiv \forall x F \land \forall x G$
2. $\exists x(F \land G) \equiv \exists x F \land \exists x G$

Solution:

1. Holds. Assume $\mathcal{A} \models \forall x(F \land G)$,
   \[ \iff \text{for all } d \in U_{\mathcal{A}}, \text{ we have } \mathcal{A}[d/x] \models F \text{ and } \mathcal{A}[d/x] \models G, \]
   \[ \iff \text{for all } d_1 \in U_{\mathcal{A}}, \text{ we have } \mathcal{A}[d_1/x] \models F \text{ and for all } d_2 \in U_{\mathcal{A}}, \text{ we have } \mathcal{A}[d_2/x] \models G \]
   \[ \iff \mathcal{A} \models \forall x F \land \forall x G \]

2. Does not hold. Let $F = P(x)$ and $G = Q(x)$, $U_{\mathcal{A}} = \{0, 1\}$, $P^{\mathcal{A}} = \{0\}$, and $Q^{\mathcal{A}} = \{1\}$.
   Clearly, $\mathcal{A} \models \exists x F \land \exists x G$ but $\mathcal{A} \not\models \exists x(F \land G)$
Exercise 6.2. [Infinite Models]
Consider predicate logic with equality. We use infix notation for equality, and abbreviate \( \neg(s = t) \) by \( s \neq t \). Moreover, we call a structure finite iff its universe is finite.

1. Specify a finite model for the formula \( \forall x (c \neq f(x) \land x \neq f(x)) \).
2. Specify a model for the formula \( \forall x \forall y (c \neq f(x) \land (f(x) = f(y) \rightarrow x = y)) \).
3. Show that the above formula has no finite model.

Solution:

1. \( U^A = \{0, 1, 2\} \subset \mathbb{N} \) and \( c^A = 0 \) and \( f^A(0) = 1 \mid f^A(n + 1) = 2 - n \)
2. \( U^A = \mathbb{N} \) and \( c^A = 0 \) and \( f^A(n) = n + 1 \)
3. Assume a model \( A \). First note that the properties transfer to the semantic level, i.e., we have for all \( x, y \in U_A \):

\[
\begin{align*}
  c^A &\neq f^A(x) \quad (1) \\
  f^A(x) &= f^A(y) \implies x = y \quad (2)
\end{align*}
\]

Now, we are in a position to show that \( U_A \) is infinite. For this, we define \( x_i = (f^A)^i(c^A) \), i.e. \( i \) times \( f^A \) applied to \( c^A \). Clearly, we have \( x_i \in U_A \) for all \( i \). We now show that \( i < j \) implies \( x_i \neq x_j \), immediately yielding infinity of \( U_A \). We do induction on \( i \). For \( 0 \), we have \( x_0 = c^A \neq f^A(\ldots) = x_j \) (by (1)). For \( i + 1 \), the induction hypothesis gives us \( x_i \neq x_j \), which implies \( x_{i+1} \neq x_{j+1} \) (by (2)). qed.

Exercise 6.3. [Skolem Form]
Convert the following formula into – in order – a rectified formula, closed and rectified formula, RPF and Skolem form.

\[
P(x) \land \forall x (Q(x) \land \forall x \exists y P(f(x, y)))
\]

Solution:

\[
\begin{align*}
P(x) \land \forall x (Q(x) \land \forall x \exists y P(f(x, y))) &\quad \text{rectified} \\
\sim P(x) \land \forall x_1(Q(x_1) \land \forall x_2 \exists y P(f(x_2, y))) &\quad \text{rectified and closed} \\
\sim \exists x P(x) \land \forall x_1(Q(x_1) \land \forall x_2 \exists y P(f(x_2, y))) &\quad \text{RPF} \\
\sim \exists x_1 \forall x_2 (P(g) \land (Q(x_1) \land P(f(x_2, h(x_1, x_2))))) &\quad \text{Skolem form}
\end{align*}
\]
Homework 6.1. [Predicate Logic] (6 points)

1. Specify a satisfiable formula $F$ such that for all models $\mathcal{A}$ of $F$, we have $|U_{\mathcal{A}}| \geq 4$. You may or may not use equality.

2. Can you also specify a satisfiable formula $F$ such that for all models $\mathcal{A}$ of $F$, we have $|U_{\mathcal{A}}| \leq 4$? Consider both predicate logic with and without equality.

Homework 6.2. [Skolem Form] (6 points)
Convert the following formulas into – in order – a rectified formula, closed and rectified formula, RPF and Skolem form.

1. $\forall x \exists y \forall z \exists w(\neg Q(f(x), y) \land P(a, w))$
2. $\forall z (\exists y (P(x, g(y), z)) \lor \neg \forall x Q(x))$

Homework 6.3. [Orders] (8 points)
A reflexive and transitive relation is called preorder. In predicate logic, preorders can be characterized by the formula

$$F \equiv \forall x \forall y \forall z (P(x, x) \land (P(x, y) \land P(y, z) \rightarrow P(x, z)))$$

1. Which of the following structures are models of $F$? Give an informal proof in the positive case and a counterexample for the negative case!
   (a) $U^A = \mathbb{N}$ and $P^A = \{(m, n) \mid m > n\}$
   (b) $U^A = \mathbb{Z} \times \mathbb{Z}$ and $P^A = \{((x, y), (a, b)) \mid a - x \leq b - y \}$
   (c) $U^A = \mathbb{R}$ and $P^A = \{(m, n) \mid m = n\}$

2. Let $Q(x, y)$ be specified as follows: $\forall x \forall y (P(x, y) \leftrightarrow Q(y, x))$. Assuming $P$ is a preorder, is $Q$ also a preorder?

3. Specify the notion of equivalence relations, that is, preorders that additionally satisfy symmetry.