Exercise 7.1.  [Herbrand Models]

Given the formula

\[ F = \forall x \forall y (P(f(x), g(y)) \land \neg P(g(x), f(y))) \]

1. Specify a Herbrand model for \( F \).
2. Specify a Herbrand structure suitable for \( F \), which is not a model of \( F \).

Solution:

We define \( U_A = T(F) \), i.e., the Herbrand universe for \( F \). We invent a constant \( a \in T(F) \). We define \( f^A \) and \( g^A \) to be the Herbrand interpretations.

1. \( P^A = \{(f(t_1), g(t_2)) \mid t_1, t_2 \in T(F)\} \).
2. \( P^A = \{(g(t_1), f(t_2)) \mid t_1, t_2 \in T(F)\} \).
Exercise 7.2. [(In)finite Models]

1. Show that any model (for a formula of predicate logic) with a universe of size $n$ can be extended to a model of size $m$ for any $m \geq n$. Can it also be extended to an infinite model?

2. Now consider the extension of predicate logic with equality. Does above property still hold?

Solution:

1. Let $\mathcal{A}$ be a model. We pick any $d \in U_{\mathcal{A}}$ as an element to “clone” $m - n$ times.

   The precise construction works as follows: We define $D = \{(d, k) \mid k \in \mathbb{N} \land k < m - n\}$. Now, we extend $U_{\mathcal{A}}$ with $D$.

   Let $\mathcal{A}'$ be a structure with the universe $U_{\mathcal{A}'} = U_{\mathcal{A}} \uplus D$. All functions and predicate symbols are interpreted identically to $\mathcal{A}$, with the extension that all elements $(d, k)$ are treated as $d$.

   We interpret a unary predicate $P$ as follows:

   $$
P_{\mathcal{A}'} = \begin{cases} 
P_{\mathcal{A}} & \text{if } d \notin P_{\mathcal{A}} \\
P_{\mathcal{A}} \cup D & \text{otherwise} 
\end{cases}
$$

   The construction can be extended for $n$-ary predicates, by looking at each position separately.

   Similarly, we can give the modified interpretation for a unary function symbol $f$:

   $$
f_{\mathcal{A}'}(x) = \begin{cases} 
f_{\mathcal{A}}(x) & \text{if } x \notin D \\
f_{\mathcal{A}}(d) & \text{if } x = (d, k) \in D 
\end{cases}
$$

   Extending to an infinite model works in exactly the same way, except for adding infinitely many copies of $d$ by dropping the $k < m - n$ condition.

2. This does not work, because the $=$ predicate allows one to distinguish between different elements.

   Counterexample: The formula $F = \forall x \forall y (x = y)$ has a trivial model $\mathcal{A}$ with cardinality 1. Obviously, there cannot be any larger model.
Exercise 7.3. [Natural Numbers and FOL]
We consider the following axioms in an attempt to model the natural numbers in predicate logic:

1. \( F_1 = \forall x \forall y (f(x) = f(y) \rightarrow x = y) \)
2. \( F_2 = \forall x (f(x) \neq 0) \)
3. \( F_3 = \forall x (x = 0 \lor \exists y (x = f(y))) \)

Give a model with an uncountable universe for:

1. \( \{F_1, F_2\} \)
2. \( \{F_1, F_2, F_3\} \)

*Hint: A set \( S \) is uncountable if there is no bijection between \( S \) and \( \mathbb{N} \).*

*Solution:*

1. \( U_A = \mathbb{R}_0^+ \), \( 0^A = 0 \), and \( f^A(x) = x + 1 \)
   
   \( f^A \) is clearly injective and there is no \( x \) such that \( f^A(x) = 0 \), because \( -1 \notin U_A \).

2. We take \( U_A \) to be the union of the positive real numbers and the non-positive whole numbers, i.e., \( U_A = \mathbb{R}_{>0} \cup \mathbb{Z}_{\leq 0} \).

Let the symbols be interpreted as follows:

\[
0^A = 0 \\
f^A(x) = \begin{cases} 
2x & \text{if } x > 0 \\
 x - 1 & \text{if } x \leq 0 
\end{cases}
\]

(a) \( f^A \) is defined as two disjoint domains that have disjoint ranges. Both domains are injective, hence the entire function is injective.

(b) \( 0 \) is not in the range of \( f^A \): For \( x > 0 \), \( f^A(x) > 0 \) and for \( x \leq 0 \), \( f^A(x) \leq -1 \).

(c) To show: \( x \neq 0 \rightarrow \exists y (x = f(y)) \).
   
   If \( x < 0 \), then \( x \leq -1 \), hence \( x = f^A(x + 1) \).
   Otherwise, \( x = f^A \left( \frac{x}{2} \right) \).
Homework 7.1. [Invalid Herbrand Models] (8 points)
Recall the fundamental theorem from the lecture: “Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff it has a Herbrand model”.

Explain “what goes wrong” if the precondition is violated: when $F$ is not closed or not in Skolem form. Describe both cases.

Homework 7.2. [Proof of the Fundamental Theorem] (6 points)
Recall the fundamental theorem: Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff it has a Herbrand model. Give the omitted proof for the base case (slide 6, $A(G) = T(G)$).

Homework 7.3. [Herbrand Models] (6 points)
Given the formula

$$F = \forall x (P(f(x)) \leftrightarrow \neg P(x))$$

1. Specify a Herbrand model for $F$.
2. Specify a Herbrand structure suitable for $F$, which is not a model of $F$. 