Exercise 11.1.  [Decidable Theories]
Let \( S \) be a set of sentences (i.e. closed formulas) such that \( S \) is closed under consequence: if \( S \models F \) and \( F \) is closed, then \( F \in S \). Additionally, assume that \( S \) is finitely axiomatizable and complete, i.e. \( F \in S \) or \( \neg F \in S \) for any sentence \( F \).

1. Give a procedure for deciding, given only the axiomatization of \( S \), whether \( S \models F \) for a sentence \( F \).

2. Can you obtain a similar result when the assumption is that the axiom system is only recursively enumerable?

Solution:

1. Let \( M \) be the set of axioms. Run resolution on \( M \land F \) and \( M \land \neg F \) in parallel. If \( F \notin S \), then \( M \land F \vdash \square \) and the first resolution terminates. If \( F \in S \), then \( M \land \neg F \vdash \square \) and the second resolution terminates.

2. Yes, by compactness. Enumerate all finite subsets of the axiom set and run resolutions in parallel.

Exercise 11.2.  [Consequence]
Show that \( Cn \) is a closure operator, i.e. \( Cn \) fulfills the following properties:

- \( S \subseteq Cn(S) \)
- if \( S \subseteq S' \) then \( Cn(S) \subseteq Cn(S') \)
- \( Cn(Cn(S)) = Cn(S) \)

Solution:
In the following, suppose \( S, S' \) are sets of \( \Sigma \)-sentences and \( F \) is \( \Sigma \)-sentence.

\[
F \in S \implies S \models F \implies Cn(S) \models F
\]

\[
F \in Cn(S) \implies S \models F \implies S' \models F \implies F \in Cn(S')
\]

From the above two: \( Cn(S) \subseteq Cn(Cn(S)) \)

\[
F \in Cn(Cn(S)) \implies Cn(S) \models F \implies (\ast) \models F \implies F \in Cn(S)
\]

We have (\ast) because \( A \models Cn(S) \) iff \( A \models S \) by definition of \( Cn \).
Exercise 11.3.  [Axiomatizations and Compactness]
Using compactness, show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if \( Cn(\Gamma) = Cn(\Delta) \) with \( \Gamma \) countable and \( \Delta \) finite, then there is a finite \( \Gamma' \subseteq \Gamma \) with \( Cn(\Gamma') = Cn(\Gamma) \).

Solution:
Claim: We can construct a finite subset \( \Gamma' \subseteq \Gamma \) that axiomatizes \( Cn(\Delta) \). In particular, \( \Gamma' \vdash \Delta \) must hold. This is equivalent to \( \Gamma', \neg \Delta \vdash \bot \).

We also know that \( \Gamma, \neg \Delta \vdash \bot \), because \( \Gamma \) axiomatizes \( Cn(\Delta) \). Hence, the infinite set of formulas \( \Gamma \cup \{ \neg \Delta \} \) is unsatisfiable. By compactness, there must be a finite subset that is unsatisfiable.

We can find this subset by enumerating all finite subsets \( \Gamma' \subseteq \Gamma \) and running resolution on \( \Gamma', \neg \Delta \).

Exercise 11.4.  [Natural Deduction]
Prove the following formula using natural deduction.

\[ \neg (\forall x (\exists y (\neg P(x) \land P(y)))) \]

Solution:

\[
\begin{align*}
\forall E \left[ \forall x \exists y (\neg P(x) \land P(y)) \right] & \quad \neg P(x_1) \land P(y_1) \quad \forall E \left[ \forall x \exists y (\neg P(x) \land P(y)) \right] \\
\exists y (\neg P(x_1) \land P(y_1)) & \quad \exists y (\neg P(y_1) \land P(y)) \\
\neg P(y_1) & \quad \neg P(y_1) \\
\forall E & \quad \exists E \\
\bot & \quad \exists I \\
\neg (\forall x (\exists y (\neg P(x) \land P(y)))) & \quad \neg I
\end{align*}
\]
**Homework 11.1. ** [Counterexamples from Sequent Calculus] (4 points)

Consider the statement $\forall x P(x) \rightarrow \neg P(f(x))$.

1. What happens when trying to prove the validity of this formula in sequent calculus?
2. How can we derive a countermodel from the proof tree?
3. Is there a smaller countermodel?

**Homework 11.2. ** [Proofs] (8 points)

Prove the following statements using natural deduction.

1. $\neg \forall x \exists y \forall z (\neg P(x, z) \land P(z, y))$
2. $\exists x (P(x) \rightarrow \forall x P(x))$

**Homework 11.3. ** [Elementary Classes] (8 points)

In this exercise, we assume that all structures and formulas share the same signature $\Sigma$.

We define the operator $Mod(S)$ that returns the class of all structures that model a set of formulas $S$. In other words, $Mod(S)$ contains all $A$ such that $A \models S$.

A class of models $M$ is said to be $\Delta$-elementary if there is a set of formulas $S$ such that $M = Mod(S)$. If $S$ is just a singleton set, i.e. there is a formula $F$ such that $S = \{F\}$, then $M$ is elementary.

Prove:

1. A class of models $M$ is elementary if and only if there is a finite set of formulas $S$ such that $M = Mod(S)$.
2. If $M$ is elementary and $M = Mod(S)$, there is a finite subset $S' \subseteq S$ such that $M = Mod(S')$. 