Foundations of Mathematics and Grundlagenkrise

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Abstract

In this paper we see how Georg Cantor tried to find a set-theoretic foundation of mathematics with his naive set theory. After a short introduction in this theory we look at three paradoxa which show that this theory is not suitable for this aim. We get a short overview of how mathematicians tried to find a way out of this dilemma and finally how Gödel showed with his incompleteness theorems that it is not possible to find a complete and consistent foundation of mathematics.

1 Introduction

The first serious attempt to approach mathematics via a system of five axioms and five postulates in order to derive facts from them was done by Euklid in in the third century B.C.
In Euklids book "The Elements" he tried to summarize all the mathematics done so far in ancient Greece. In order to get an idea how Euklid worked we give some examples of axioms and postulates Euklid used as translated by Thomas Heath in [H^{+}56]:

- Axiom 1: Things that are equal to the same thing are equal to each other.
- Axiom 2: If equals are added to equals, then the wholes are equal.

"Let the following be postulated:

- To draw a straight line from any point to any point.
- To extend a finite straight line continuously in a straight line.
- To describe a circle with any centre and distance, the radius.
- That all right angles are equal to one another.
- (The so-called parallel postulate)That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

Of course Euklids approach was rather geometric.
But in the end we will see that this system – when leaving the fifth postulate – is a quite interesting example for a non-complete formal system since the parallel postulate is neither neccessary nor in contradiction to the other axioms and postulates.
In the late nineteenth century Georg Cantor also tried to find a axiomatic foundation of all known mathematics – the so-called naive set theory.
As we will see he failed, since his system of axioms led to paradoxa which led to the so-called fundamental crisis of mathematics. But this was not the first time when the existing believes in mathematics were unsettled by new knowledge. For this we also find an example from ancient Greece.
When you look at a square with side-length 1, you easily see that the diagonal has length $\sqrt{2}$. Before Pythagoras proved that this number $\sqrt{2}$ is irrational, the mathematicians thought that any number that occurs is a ratio of integers. But even if this was a quite surprising result, it seems that this was not as shocking as the fundamental crisis arising from the paradoxa in Cantor's naive set theory. For more information on this see e.g. [Bur62], p. 431-440, or [Zhm97], p. 170-175.

2 The foundational crisis

In this first section we will see how Georg Cantor established set theory in an axiomatic manner. We will also discover how mathematicians like Bertrand Russell found problems in the form of contradiction that occur as a consequence of Cantor's axioms. All the facts presented in this chapter can be found in chapter two of [Rob15].

2.1 Cantor's naive set theory

One of the main concepts of this chapter is the mathematical concept of a "set". Looking up the noun set in a dictionary, one finds explanations like

- A group of similar things that belong together in some way, or
- In mathematics, a set is a group of objects with stated characteristics.

To work with the term set in mathematics, we need a precise definition. In 1895, Cantor stated a quite natural way of the term set:

Definition 2.1.1. A set is any collection of definite, distinguishable objects of our intuition or of our intellect to be conceived as a whole (i.e., regarded as a single unity).

So, a set can contain basically anything that can be distinguished: For example there is a set containing one specific tree, the number one and this paper. But there are also weird sets, like the set containing only itself or the set of all sets, which also contains itself. In this chapter we will actually see, that the set of all sets involves one of the announced contradictions.
According to Robic, there are three principles which Cantor used like axioms:

Axiom 2.1.2. A set is completely determined by its members.

This essentially means, that we can specify a set by listing all members of it. Usually this is done by writing the members in between curly braces like for example

$$\{x, y, z\}$$

for the set containing the elements x,y and z or writing

$$\{x|x \text{ has property } P\}$$

for the set of all x, that have the property P, e.g.

$$\{x|x \in \mathbb{R} \text{ and } x^2 > 4\}$$

for the set of all real numbers smaller than -2 or bigger than 2. Cantor also was not very picky concerning P:

Axiom 2.1.3. Every property determines a set.

At last, Cantor believed in the so-called "axiom of choice", a famous issue of dispute among mathematicians:

Axiom 2.1.4. Given any set F of nonempty pairwise disjoint sets, there is a set that contains exactly one member of each set in F.

Now that we have defined the term *set*, there are of course many natural possibilities to work with set: As usual Cantor also defined actions on sets like intersection, union, complement, et cetera as well as functions between sets and the properties injective, surjective and bijective.

2.2 Three paradoxa

As mentioned, this way of defining the term set is – except for the axiom of choice – quite natural and seems like a solid basis for mathematics. But we will see in this chapter how to unsettle the whole theory.
The first logical paradox was discovered in 1897 by Cesare Burali-Forti. In order to state it, we need the concept of ordinal numbers. For the first paradox, we leave out some details as they go beyond the scope of this paper.
We start with a definition.

Definition 2.2.1. Let $\mathcal{A}$ be a set. We say that $\prec$ is a linear ordering on $\mathcal{A}$, if

- $a \nprec a$
- $a \neq b$ implies $a \prec b$ or $b \prec a$.

We say that two linear ordered sets $\mathcal{A}$ and $\mathcal{B}$ are similar if there is a bijection preserving the linear orders on the two sets.
For a set $\mathcal{A}$, we call the equivalence class of all linear ordered sets that are similar to $\mathcal{A}$ the ordinal number of $\mathcal{A}$.

One can also define an order on the ordinal numbers and it is possible to show that there is no biggest ordinal number.
Now we can state the paradox of Burali-Forti: Define

$$\mathcal{A} = \{x | x \text{ is ordinal number for some set}\}.$$

Now, one can show, that the ordinal number of $\mathcal{A}$ is bigger than itself which yields a contradiction.

The second paradox was found by Cantor himself in 1899. For this we need the concept of cardinal numbers.
Roughly speaking, the cardinal number of a set $\mathcal{A}$ should be something like the number of elements of $\mathcal{A}$. Let us make this precisely:

Definition 2.2.2. Let $\mathcal{A} = \{1, ..., n\}$. We say $\mathcal{A}$ has cardinality n.
Two sets $\mathcal{A}$ and $\mathcal{B}$ have the same cardinality, if there is a bijective function $f : \mathcal{A} \to \mathcal{B}$.
If there is a surjective function $g : \mathcal{A} \to \mathcal{B}$ we say that the cardinality of $\mathcal{A}$ is bigger or equal to the cardinality of $\mathcal{B}$.
It is bigger, if it is bigger or equal but not equal.
Moreover, we say the set

$$\mathbb{N} = \{x : x \text{ is a natural number}\}$$

of natural numbers has cardinality $\aleph_0$.

It is time for a little mathematical interlude, let us prove a nice little fact.

Theorem 2.2.3. Let $\mathcal{A}$ be a non-empty set, and denote by $2^{\mathcal{A}}$ its powerset, i.e. the set of all subsets of $\mathcal{A}$. Then the cardinality of $2^{\mathcal{A}}$ is bigger than the cardinality of $\mathcal{A}$.

Proof. We need to show two things.
Firstly we show, that the cardinality of $2^{\mathcal{A}}$ is bigger or equal than the cardinality of $\mathcal{A}$. Fix x_0 in $\mathcal{A}$. But then we find a surjective function $f : 2^{\mathcal{A}} \to \mathcal{A}$ by mapping sets of the form $\{x\}$ to x, and all the other sets to x_0. Since $2^{\mathcal{A}}$ contains all subsets of $\mathcal{A}$, this map is in fact surjective.
Now we still have to argue that there cannot be a surjective map from $\mathcal{A}$ to $2^{\mathcal{A}}$. Let $g : \mathcal{A} \to 2^{\mathcal{A}}$ be any function. Consider the set

$$\mathcal{B} = \{x \in \mathcal{A} : x \notin g(x)\}.$$

Assume that there is a $y \in \mathcal{A}$ with $g(y) = \mathcal{B}$.
By the definition of $\mathcal{B}$, $y \in \mathcal{B}$ would imply $y \notin g(y) = \mathcal{B}$, a contradiction.
Moreover, $y \notin g(y) = \mathcal{B}$ would imply $y \in \mathcal{B}$, as it fulfills the required property to be in $\mathcal{B}$.
Hence this gives a contradiction. Thus there is no surjective function and therefore no such bijective function. □

With this result we can understand the second paradox. Let us denote by $\mathcal{A}$ the set of all sets, an object we found strange from the beginning on.
It is clear, that all subsets of $\mathcal{A}$ are again sets, so the set $2^{\mathcal{A}}$ also consists of sets. By mapping every set in $\mathcal{A}$ to itself, if it is contained in $2^{\mathcal{A}}$, and to the empty set if not, we obtain a surjective map $f : \mathcal{A} \to 2^{\mathcal{A}}$ which contradicts the previous theorem.

The third paradox is probably the best known and was found by Bertrand Russell in 1901. It makes use of the fact that any property defines a set.
Consider the set

$$\mathcal{A} = \{\mathcal{S} | \mathcal{S} \text{ is a set } \wedge \mathcal{S} \notin \mathcal{S}\}.$$

It is a natural question to ask if $\mathcal{A}$ contains itself.
If so, we have $\mathcal{A} \subset \mathcal{A}$ which violates the property defining $\mathcal{A}$, hence $\mathcal{A} \notin \mathcal{A}$.
Since this is a contradiction we must have $\mathcal{A} \notin \mathcal{A}$. But then $\mathcal{A}$ fulfills the property defining $\mathcal{A}$, hence $\mathcal{A} \in \mathcal{A}$, which is also a contradiction.

2.3 A consequence of the downfall of naive set theory

Since the theory of naive set theory had some serious disagreements with itself, it was not further suitable as a foundation of mathematics.
The german mathematician Gottlob Frege used Cantor's naive set theory as foundation of his book "Grundlagen der Arithmetik" ("Foundations of arithmetic") in which he tried to trace back terms like "number" to logic and set theory which was published in 1893. When Russell pointed out to him that this theory of sets was not consistent, he retired from his occupation with logic.

3 Solution trials for the foundational crisis – The schools of recovery

Not only Gottlob Frege was shocked by the failure of the naive set theory. In order to get to a different foundation of mathematics, three main schools were developed by the leading mathematicians of this time.

As in the last chapter, the following is taken from chapter two of [Rob15]. Further information on logicism can be found in [Car31], on intuitionism in [Hey31] and on formalism in [Neu31]. These three articles give a quite genuine insight on these three theories as they have been written in 1931 by representatives of these schools.

3.1 Intuitionism

The main representatives of Intuitionism were the Dutch mathematician Luitzen Brouwer and his student Arend Heyting. As the name suggests, Intuitionism sees mathematical objects as a part of the intuition, as a part of the mind. We will give two examples of remarkable convictions that went along with this point of view.

Firstly, we remind that Cantor allowed any property to define a set. In particular, he considered infinite sets as the set of natural numbers $\mathbb{N}$ as actually existing. The intuitionists were rather critical about this: They thought of infinite sets as potentially infinite. In an infinite set we thus can find any number of elements, but we cannot work with all of them as infinitely many.

Let us see a second restriction the intuitionists gave themselves: Cantor believed in the principal "tertium non datur" ("there is no third"), which says: A statement is either true or false, there is no third possibility. The intuitionists saw no reason, why this should be natural and did not make use of this principle.

We demostrate how these two restrictions can prevent someone from proving a statement: Let $\mathcal{A}$ be an infinite set, P a property which any element of $\mathcal{A}$ can have. We ask if the following is true:

$$\forall x \in \mathcal{A} : x \text{ has property } P.$$

Let us also assume that we are unable to find a direct proof, i.e. a reason, why P must be true for all elements of $\mathcal{A}$. Since $\mathcal{A}$ is not finite, we can not check the property P for every single element of $\mathcal{A}$. The next way would be to prove the statement by contradiction: This starts by assuming

$$\neg\forall x \in \mathcal{A} : P(x)$$

and leading this to a contradiction. Because of the missing of "tertium non datur", we can not deduce from this that the opposite is true.

Thus, if we cannot find a direct prove, we get stuck and the answer to our question is not clear. (In fact, for an intuitionst it is not even clear that it is either true or false, but maybe something else.)

3.2 Logicism

The school of Logicism already started before the inconsistencies in Cantor's naive set theory were discovered. One of the reprensentatives was Gottlob Frege of which we already read before. Other well-known representatives of this stream were Boole, Peano, Russell, and Whitehead.

As before, the name of this school already suggests the mathematical approach: Its main goal was to build the whole theory of mathematics one pure logic. From 1910 to 1913 Whitehead and Russell described this approach in their book "Principia Mathematica". But even though Principia Mathematica avoided all known paradoxa, none of these mathematicians was able to prove consistency or completenes of Principia Mathematica, i.e. freeness of paradoxa and the possibility to prove every true statement in this setting.

But even though the representatives of the school of logicism failed by inventing a pure logical foundation of mathematics, they developed powerful tools which are used till today: Beside important logical concepts like "First Order Logic", Peano was the first one using logical symbols like "$\in$" or "$\Rightarrow$".

3.3 Formalism

The basic idea of Formalism is the contra-position to the convictions of the intuitionists: The school of Formalism worked with mathematic by using symbols and syntax in a purely abstract way. The main initiator of this theory was David Hilbert.
For the formalists, every mathematical statement was a finite sequence of symbols or language and is completely detached from the real world. The formalists worked with so-called *formal systems*:
A formal systems consists of a finite set of symbols for constructing formulas which are finite sequences of symbols, a decision procedure, which can decide whether a formula is well formed or not, some formulas assumed to be true, which are called axioms and a set of interference rules which tell one how to build sequences of formulas. The formalists goal was to find a complete and consistent such system of mathematics.
The main achievements of the school of formalism were the achievements of the "Hilbertprogramm" and from which Ernst Zermelo and A.A. Fraenkel developed the Zermelo Fraenkel set theory, which is used a lot in modern mathematics.

None of the three schools reached their main goal – to find a complete and consistent foundation for mathematics – in the next chapter we will see, why this failure was inevitable.

4 The last foundational crisis: Gödel's incompleteness theorems

In autumn 1930 the Austrian mathematician Kurt Gödel attendet a conference in Königsberg, Germany, where Rudolf Carnap refered on logicism, Arend Heyting on intuitionism and Johann von Neumann on formalism. Reading their speaches, on easily notices that their tone was way more reconciliatory than the disputes from the preceeding twenty years would suggest. For more information on this conference see for example [Pas].
On the last day of this conference in a discussion between Carnap, von Neumann, Scholz and Gödel, Kurt Gödel also presented the topic of his dissertation, in which he worked on the completeness of first order logic.
But at this time Kurt Gödel already had discovered his first incompleteness theorem which explains, why mathematicians were unable to find a way out of incompleteness and inconsistencies.
The facts from this chapter are taken from [Ste13], Gödel's original article can be found in [Göd31] To understand Gödel's statements we need to explain some terms.

Definition 4.0.1. A *formal system* again consists of a *language* with specified well-defined statements, some axioms and some interference rules.
A *formal system* $\mathcal{T}$ is called consistent, if there is no statement A, such that both A and its negation follows from $\mathcal{T}$.
A system is called complete, if for every well-defined statement A either A or its negation follows from $\mathcal{T}$.

These two definitions are quite natural. In addition to them, Gödel defined a property called "sufficiently powerful" which essentially describes if the system is big enough to describe basic mathematical concepts like natural numbers and a property called recursively enumerable which in some sense excludes infinite long proofs.
Now Gödel's incompleteness theorem states the following:

Theorem 4.0.2. Any sufficiently powerful, recursively enumerable formal system is either inconsistent or incomplete.

His second theorem is also interesting:

Theorem 4.0.3. Any sufficiently powerful consistent formal system cannot prove its own consistency.

The first theorem of Gödel basically shows that the efforts of all three school to find a complete and were forlorn. Well known examples for incomplete formal systems are the Zermelo Fraenkel set theory without the axiom of choice since this does not follow from the other axioms.
Another prominent example are the first four axioms in Euklid's Elements, which do not imply the parallel axiom.
Of course the theorems of Gödel have more severe consequences for mathematicians than Russell paradox as the latter one only refuses only the consistency of one formal system while Gödel shows that there are no suitable systems to build the foundations of mathematics, but at least it gives an answer on the question about the existence of such a system.

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