HOL Foundations

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HOL Foundations

- HOL is a family of proof assistants, using a variant of higher-order logic
- HOL4 is the primary descendent, still being actively developed on: https://hol-theorem-prover.org/
- HOL is the predecessor of Isabelle
- HOL has its roots in the LCF formalism

LCF formalism

- In 1969, the LCF ('Logic for computable functions') formalism was devised by Dana Scott
- Intention: Improved reasoning about recursively defined functions in denotational semantics
- Denotational semantics deals with finding mathematical objects ('domains') to explain the behavior of computer programs
- Published in 1993

The language of the LCF formalism

- Terms: Typed λ-terms; i.e. either variables, constants, λ-abstractions or λ-applications
- Formulae: Predicate calculus
- Types: Scott Domains

Stanford LCF

- In 1972, Milner, Diffie, Weyhrauch and Newey developed the proof-checker LCF at Stanford University
- It was based on the LCF formalism

Features of Stanford LCF

- ”The proof-checking program is designed to allow the user interactively to generate formal proofs about computable functions and functionals over a variety of domains, including those of interest to the computer scientist for example, integers, lists and computer programs and their semantics. The user’s task is alleviated by two features: a subgoaling facility and a powerful simplification mechanism.“ (Robin Milner)

Shortcomings of Stanford LCF

- Storage of proofs filled up memory quickly
- Repertoire of proof commands was immutable

Edinburgh LCF

- In Edinburgh, Milner tackled the problems of Stanford LCF
- Only result of proofs, not proofs themselves, should be stored
- For full customizability, Milner developed a strictly typed programming language ML ('Meta-Language')

Features of ML

- Exception handling mechanism
- Novel polymorphic type system (a term with type variables is a single polymorphic term)
- Own abstract data type for theorems

$\Longrightarrow$ All theorems must have been correctly deduced simply because of their type

Tactics

- A tactic is a function with

  - Input: Goal, that needs to be proven

  - Output: List of sub-goals along with a justification function

- Notation:

$$\frac{goal}{goal_1 \;\; goal_2 \;\; \ldots \;\; goal_n}$$

- Example: (induction)

$$\frac{\forall n.t[n]}{t[0] \quad \{t[n]\}t[SUC\ n]}$$

Tacticals

- A tactical is a function, that can compose tactics and returns a tactic.
- Example:

  - Let S and T be tactics and 'THEN' a tactical. Then 'S THEN T' applies S to some goal and then applies T to all sub-goals produced by S

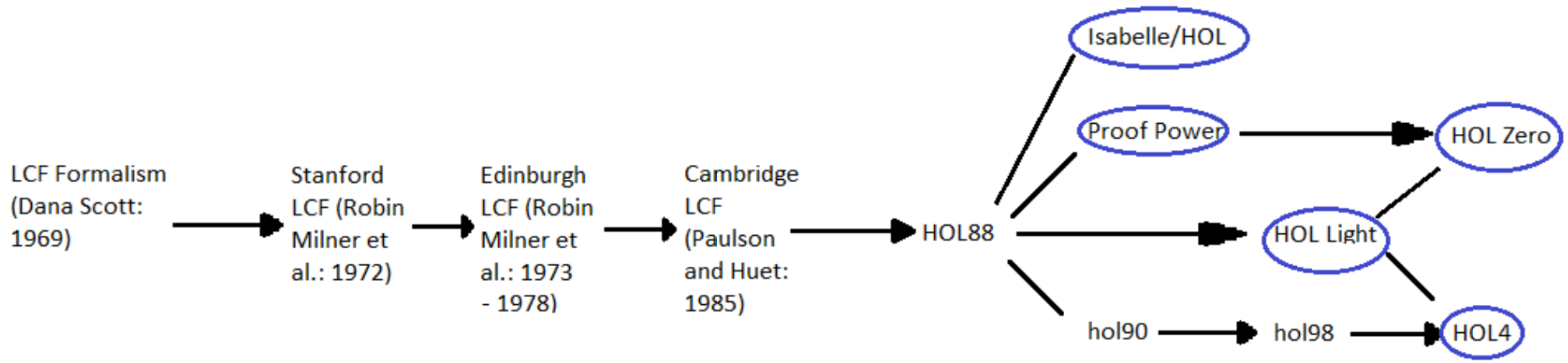
Cambridge LCF

- Gerard Huet ported Edinburgh LCF to the Lisp dialects Le Lisp and MacLisp
- Larry Paulson then improved Huet's code
- Many features and techniques were added
- The resulting system was called Cambridge LCF due Paulson's workplace and got ported to Standard ML

HOL

- Mike Gordon – inspired by a theorem proved by Robin Milner – invented a notation called LSM ('Logic of sequential machines')
- Gordon's main interest was the formal verification of hardware
- He then combined LSM with a version of Cambridge LCF, encoded terms in predicate calculus, which resulted in HOL
- Gordon used higher-order logic to be able to adequately model hardware

From LCF to HOL



HOL's logic and novelties

- The language corresponds to that of the LCF formalism with the difference, that types were interpreted as sets instead of Scott Domains
- Higher-order logic admits quantification over sets or predicates, that are nested arbitrarily deep
- Example of a third-order term:

$$\forall Q \exists R \in Q \exists f \exists x \exists y : \; R(f(x)) \to R(y)$$

- Two theories form the basis of HOL (bool, ind)

The theory *bool*

- Contains:
  - Primitive type 'bool'
  - Four axioms for higher-order logic
  - Three primitive constants (Equality, Implication and Choice) and some more useful but less important constants
- With these three constants we can define $\top$ (truth), $\bot$ (falsity), $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\forall$ (universal quantification), $\exists$ (existential quantification) and $\exists!$ (unique existence quantification)

The Choice- or Hilbert's ε-operator

- Let t[x] be a term of type σ → bool with a free variable x
- εx.t[x] returns some a in σ, such that t[a] is true. If t[a] is false for all a in σ, then εx.t[x] denotes some unspecified element in σ
- With the Hilbert-operator, we implicitly implement the Axiom of Choice

Examples

- $\varepsilon n.n < 5$ denotes some unspecified number below 5
- $\varepsilon n.(n^2 = 25) \wedge (n \geq 0)$ denotes 5
- $\varepsilon n.\neg(n = n)$ is some unspecified number

Four axioms in *bool*

$$\vdash \forall b.\ (b = \top) \lor (b = \bot)$$
$$\vdash \forall b_1\ b_2.\ (b_1 \Rightarrow b_2) \Rightarrow (b_2 \Rightarrow b_1) \Rightarrow (b_1 = b_2)$$
$$\vdash \forall f.\ (\lambda x.\ f\ x) = f$$
$$\vdash \forall P\ x.\ P\ x \Rightarrow P(\$\varepsilon\ P)$$

The theory *ind*

- Contains:
  - Primitive type 'ind' (individuals)
  - Axiom of Infinity:

$$\vdash \exists f : ind \to ind.\ (\mathbf{One\_One}\ f) \wedge \neg(\mathbf{Onto}\ f)$$

- The Axiom of Infinity asserts that ind denotes an infinite set (would be an impossible construction in bool)
- Axioms of bool and ind sufficient for developing standard mathematics

Inference rules in HOL

- HOL uses eight inference rules:
  - ASSUME: Assumption Introduction
  - REFL: Reflexivity
  - BETA_CONV: Beta-conversion
  - SUBST: Substitution
  - ABS: Abstraction
  - INST_TYPE: Type Instantiation
  - DISCH: Discharging an assumption
  - MP: Modus Ponens

Two inference rules

- DISCH:

$$\frac{\Gamma \vdash t_2}{\Gamma - \{t_1\} \vdash t_1 \Rightarrow t_2}$$

- BETA_CONV:

$$\frac{}{\vdash (\lambda x.t_1)t_2 = t_1[t_2/x]}$$

The LCF approach in ML

- Logical inference rules are implemented as functions
- Modus Ponens as an example:

$$\frac{\Gamma \vdash p \Rightarrow q \qquad \Delta \vdash p}{\Gamma \cup \Delta \vdash q}$$

- In ML: $\mathbf{val\ MP{:}thm} \rightarrow \mathbf{thm} \rightarrow \mathbf{thm}$

$$\mathrm{MP}(\Gamma \vdash p \Rightarrow q)(\Delta \vdash p) = (\Gamma \cup \Delta \vdash q)$$

HOL and Set theory - Comparison

- HOL fundamentally bases on typed higher-order logic, more generally on type theory

HOL and Set theory - Comparison

Type Theory	**Set Theory**
- No standard formulation for typed higher-order logic - Functions as most basic operators, in simply typed lambda calculus even the only type operator - Natural numbers defined as inductive type with two constructors: $1: \mathbb{N}$, $S: \mathbb{N} \to \mathbb{N}$ - Easy access to tools for indexing terms, structuring data, checking types - Proofs/Theorems often shorter and simpler - Not difficult to build set theory on top of type theory. - Elements can usually belong to only one type	- ZFC is the foundation for mathematics as recognized by most mathematicians. - Natural numbers defined as nested sets of the empty set: $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\}$ - Known to most mathematicians - Elements can belong to different sets at the same time

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