A Purely Definitional Universal Domain (Draft)

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Abstract. Existing theorem prover tools do not adequately support reasoning about general recursive datatypes. Better support for such datatypes would facilitate reasoning about a wide variety of real-world programs, including those written in continuation-passing style, that are beyond the scope of current tools.
This paper introduces a new formalization of a universal domain that is suitable for modeling general recursive datatypes. The construction is purely definitional, introducing no new axioms. Defining recursive types in terms of this universal domain will allow a theorem prover to derive strong reasoning principles, with soundness ensured by construction.

1 Introduction

One of the main attractions of pure functional languages like Haskell is that they promise to be easy to reason about. However, that promise has not yet been fulfilled. To illustrate this point, let us define a couple of datatypes and functions, and try to prove some simple properties.

```
data Cont r a = MkCont ((a -> r) -> r)

mapCont :: (a -> b) -> Cont r a -> Cont r b
mapCont f (MkCont c) = MkCont (\k -> c (k . f))

data Resumption r a = Done a | More (Cont r (Resumption r a))

bind :: Resumption r a -> (a -> Resumption r b) -> Resumption r b
bind (Done x) f = f x
bind (More c) f = More (mapCont (\r -> bind r f) c)
```

Haskell programmers may recognize type `Cont` as a standard continuation monad. Along with the type definition is a map function `mapCont`, for which we expect the functor laws to hold. By itself, type `Cont` is not difficult to work with. None of the definitions are recursive, so they can be formalized easily in most any theorem prover. Proofs of the functor laws `mapCont id = id` and `mapCont (f . g) = mapCont f . mapCont g` are straightforward.

Things get more interesting with the next datatype definition. Monad experts might notice that type `Resumption` is basically a resumption monad transformer wrapped around a continuation monad. The function `bind` is the monadic bind operation for the `Resumption` monad; together with `Done` as the monadic unit, we should expect `bind` to satisfy the monad laws.

The first monad law follows trivially from the definition of `bind`. Instead, let's consider the second monad law (also known as the right-unit law) which states that `bind r Done = r`. How can we go about proving this, formally or otherwise?

It might be worthwhile to try case analysis on `r`, for a start. If `r` is equal to `Done x`, then from the definition of `bind` we have `bind (Done x) Done = Done x`, so the law holds in this case. Furthermore, if `r` is equal to ⊥, then from the strictness of `bind` we have `bind ⊥ Done = ⊥`, so the law also holds for ⊥. Finally, we must consider the case when `r` is equal to `More c`. Using the definition of `bind` we obtain the following:

```
bind (More c) Done = More (mapCont (\r -> bind r Done) c)
```

Now, if we could only rewrite the `bind r Done` on the right-hand side to `r`, then we could use the functor identity law for `mapCont` to simplify the entire right-hand side to `More c`.

Maybe if we had used an appropriate induction rule, then we could have used an inductive hypothesis to justify rewriting `bind r Done` to `r`. However, this would have to be a rather unusual induction rule. With more ordinary datatypes, the inductive hypothesis simply assumes that the property being proved holds for an immediate subterm. For example, when doing induction over lists, we get to assume `P(xs)` in order to show `P(x : xs)`. This kind of inductive hypothesis will not work for type `Resumption`, because of the indirect recursion in its definition.

In fact, an induction rule for `Resumption` appropriate for our proof does exist, and it is indeed rather unusual. (The proof of the second monad law using this induction scheme is left as an exercise for the reader.)

$$\frac{\begin{array}{l}\texttt{admissible(P)}\\ \texttt{P(undefined)}\\ \forall\texttt{x. P(Done x)}\\ \forall\texttt{f c. (}\forall\texttt{x. P(f x))} \longrightarrow \texttt{P(More (mapCont f c))}\end{array}}{\forall\texttt{x. P(x)}} \qquad (1)$$

Possible questions to ask are, "Where did this rule come from?" and "How can we be sure that this rule is correct?" Many readers are probably also wondering, "How did `mapCont` get in there?" But for an implementer of theorem proving tools, the most important question is, "How can we produce similar induction rules for other datatypes?" We will come back to these questions later on.

Unfortunately, a fully mechanized semantics of general recursive datatypes does not yet exist. Various theorem provers have facilities for defining recursive datatypes, but none can properly deal with datatype definitions like the `Resumption` type introduced earlier. The non–strictly positive recursion causes the definition to be rejected by both Isabelle/HOL's datatype package and Coq's inductive definition mechanism.

Of all the currently available theorem proving tools, the Isabelle/HOLCF domain package is the closest to being able to support such datatypes. Since it uses the continuous function space, it is not limited to strictly positive recursion. However, the HOLCF domain package also has some shortcomings. For example, it does not yet support indirect recursion properly—it produces induction rules that are too weak.

Many of its problems stem from the fact that it generates non-trivial axioms "on the fly": For each type definition, the domain package declares the existence of the new type (without defining it), and asserts an appropriate type isomorphism and induction rule. The most obvious worry with this design is the potential for unsoundness. For example, early versions of the domain package could be made to assert a type isomorphism between a type and its powerset. On the other hand, the desire to avoid unsoundness can lead to an implementation that is too conservative.

In contrast with the current domain package, the Isabelle/HOL inductive datatype package [13] is purely definitional. It uses a parameterized universe type, of which new datatypes are defined as subsets. Induction rules are not asserted as axioms; rather, they are proved as theorems. Using a similar design for the HOLCF domain package would allow strong reasoning principles to be generated, with soundness ensured by construction.

The original contributions of this paper are as follows:

– A new variation on the construction of a universal domain that can represent a wide variety of types, including sums, products, continuous function space, powerdomains, and recursive types built from these. Universal domain elements are defined in terms of sets of natural numbers, using ideal completion—thus the construction is suitable for simply-typed, higher-order logic theorem provers.
– A formalization of this construction in the HOLCF library of the Isabelle theorem prover. The formalization is fully definitional; no new axioms are asserted.

Section 2 reviews various domain theory concepts used in the HOLCF formalization. The construction of the universal domain type itself, along with embedding and projection functions, are covered in Section 3. Then Section 4 describes how the universal domain can be used to implement a datatype definition package. After a discussion of related work in Section 5, conclusions and directions for future work are found in Section 6.

2 Background Concepts

HOLCF [11] is a library of domain theory built on top of the Isabelle/HOL theorem prover. HOLCF defines many standard notions like complete partial orders and continuity; it also defines standard type constructors like the continuous function space, and strict sums and products. The remainder of this section is devoted to some more specialized concepts from HOLCF that support the formalization of the universal domain.

2.1 Embedding-Projection Pairs

Some types can be embedded within other types. The concept of an *embedding-projection pair* (often shortened to *ep-pair*) formalizes this notion. Let A and B be cpos, and $e : A \to B$ and $p : B \to A$ be continuous functions. Then e and p are an ep-pair if $p \circ e = \mathrm{Id}_A$ and $e \circ p \sqsubseteq \mathrm{Id}_B$. The existence of such an ep-pair means that type A can be embedded within type B.

```
data Shrub = Node Shrub Shrub | Tip
data Tree = Branch Tree Tree | Leaf | Twig

embed :: Shrub -> Tree
embed (Node l r) = Branch (embed l) (embed r)
embed Tip = Twig

project :: Tree -> Shrub
project (Branch l r) = Node (project l) (project r)
project Leaf = undefined
project Twig = Tip

deflate :: Tree -> Tree
deflate (Branch l r) = Branch (deflate l) (deflate r)
deflate Leaf = undefined
deflate Twig = Twig
```

Fig. 1. Embedding-projection pairs and deflations in Haskell. Function `deflate` is equal to the composition of functions `embed` and `project`.

Figure 1 shows an example in Haskell, where the the type `Shrub` is embedded into the larger type `Tree`. If we embed a value from type `Shrub` into type `Tree`, and then project back out, then we always get back the same value. In other words, for all `s` of type `Shrub`, we have `project (embed s) = s`. On the other hand, if we start with `t` of type `Tree`, project out to type `Shrub`, then embed back into type `Tree`, we may or may not get back the same value we started with. If `t` contains no `Leaf` constructors at all, then we have `embed (project t) = t`. Otherwise, what we end up with is basically a tree with all its leaves stripped off—each `Leaf` constructor is replaced with $\bot$.

2.2 Deflations

Cpo types may contain other cpos as subsets. A *deflation*[1] is a way to encode such a sub-cpo as a continuous function. Let B be a cpo, and $d : B \to B$ be a

[1] My usage of *deflation* follows Gunter [4]. Many authors use the term *projection* to refer to the same concept, but I prefer *deflation* because it avoids confusion with the second half of an ep-pair.

continuous function. Then d is a deflation if $d \circ d = d \sqsubseteq \mathrm{Id}_B$. The image set of deflation $d : B \to B$ gives a sub-cpo of B.

Essentially, a deflation is a *value* that represents a *type*. For example, the function `deflate` in Fig. 1 is a deflation; its image set consists of exactly those values of type `Tree` that contain no `Leaf` constructors. Note that while the the definition of `deflate` does not mention type `Shrub` at all, its image set is isomorphic to type `Shrub`: In other words, `deflate` (a function value) is a representation of `Shrub` (a type).

While types can be represented by deflations, type *constructors* (which are like functions from types to types) can be represented as functions from deflations to deflations. For example, the `map` function represents Haskell's list type constructor: While `deflate` is a deflation on type `Tree` that represents type `Shrub`, `map deflate` is a deflation on type `[Tree]` that represents type `[Shrub]`.

Deflations and ep-pairs are closely related. Given an ep-pair (e, p) from cpo A into cpo B, the composition $e \circ p$ is a deflation on B whose image set is isomorphic to A. Conversely, every deflation $d : B \to B$ also gives rise to an ep-pair. Define the cpo A to be the image set of d; also define e to be the inclusion map from A to B, and define $p = d$. Then (e, p) is an embedding-projection pair. So saying that there exists an ep-pair from A to B is equivalent to saying that there exists a deflation on B whose image set is isomorphic to A.

Finally we are ready to talk about what it means for a cpo to be a universal domain. A cpo U is universal for a class of cpos, if for every cpo D in the class, there exists an ep-pair from D into U. Equivalently, for every D there must exist a deflation on U with an image set isomorphic to D.

2.3 Algebraic and Bifinite Cpos

Lazy recursive datatypes often have infinite as well as finite values.[2] For example, we can define a datatype of recursive lazy lists of booleans:

```
data BoolList = Nil | Cons Bool BoolList
```

Finite values of type `BoolList` include total values like `Cons False Nil`, and `Cons True (Cons False Nil)`, along with partial finite values like `Cons False undefined`. On the other hand, recursive definitions can yield infinite values:

```
trues :: BoolList
trues = Cons True trues
```

One way to characterize the set of finite values is in terms of an `approx` function, defined below. The function `approx` is similar to the standard list function `take` that we all know and love, except that `approx 0` returns $\bot$ instead of `Nil`. (This makes each `approx n` into a deflation.) A value `xs` of type `BoolList` is finite if and only if there exists some `n` such that `approx n xs = xs`.

[2] *Compactness* is the precise technical version of the more intuitive concept of *finiteness*. The concepts do not necessarily coincide: For example, in a cpo of ordinals, $\omega + 1$ is compact but not finite. In the context of recursive datatypes, however, the concepts are equivalent.

```
approx :: Int -> BoolList -> BoolList
approx 0 xs = undefined
approx n Nil = Nil
approx n (Cons x xs) = Cons x (approx (n-1) xs)
```

The function `approx` is so named because for any input value `xs` it generates a sequence of finite approximations to `xs`. For example, the first few approximations to `trues` are $\bot$, `Cons True` $\bot$, `Cons True (Cons True` $\bot$`)`, and so on. Each is finite, but the least upper bound of the sequence is the infinite value `trues`. This property of a cpo, where every infinite value can be written as the least upper bound of a chain of finite values, is called *algebraicity*. Thus `BoolList` is an *algebraic cpo*.

The sequence of deflations `approx n` is a chain of functions whose least upper bound is the identity function. In terms of image sets, we have a sequence of partial orders whose limit is the whole type `BoolList`.

A further property of `approx` which may not be immediately apparent is that for any `n`, the image of `approx n` is a finite set. This means that image sets of `approx n` yield a sequence of *finite* partial orders. As a limit of finite partial orders, we say that type `BoolList` is a *bifinite* cpo. More precisely, as a limit of *countably many* finite partial orders, `BoolList` is an *omega*-bifinite cpo.[3]

The omega-bifinites are a useful class of cpos because bifiniteness is preserved by all of the type constructors defined in HOLCF. Furthermore, all Haskell datatypes are omega-bifinite. Basically any type constructor that preserves finiteness will preserve bifiniteness as well. More details about the formalization of omega-bifinite domains in HOLCF can be found in [8].

2.4 Ideal Completion and Continuous Extensions

In an algebraic cpo the set of finite elements, together with the ordering relation on them, completely determines the structure of the entire cpo. We say that the set of finite elements forms a *basis* for the cpo, and the entire cpo is a *completion* of the basis.

Given a basis B with ordering relation $(\preceq)$, we can reconstruct the whole algebraic cpo. The standard process for doing this is called *ideal completion*, and it is done by considering the set of ideals over the basis.

Definition 1. *A set S is an* ideal *with respect to partial preorder relation $(\preceq)$ if it has the following properties:*

- *S is nonempty: $\exists x.\, x \in S$*
- *S is downward-closed: $\forall x\, y.\, x \preceq y \longrightarrow y \in S \longrightarrow x \in S$*
- *S is directed (i.e. has an upper bound for any pair of elements):* $\forall x\, y.\, x \in S \longrightarrow y \in S \longrightarrow (\exists z.\, z \in S \wedge x \preceq z \wedge y \preceq z)$

A principal ideal *is an ideal of the form $\{y \mid y \preceq x\}$ for some x, denoted $\downarrow x$.*

[3] "SFP domain" is another name, introduced by Plotkin [14], that is used for the same concept—the name stands for Sequence of Finite Posets.

The set of all ideals over $\langle B, \preceq \rangle$ is denoted $\mathrm{Idl}(B)$; when ordered by subset inclusion, $\mathrm{Idl}(B)$ forms an algebraic cpo. The compact elements of $\mathrm{Idl}(B)$ are exactly those represented by principal ideals.

Note that the relation $(\preceq)$ does not need to be antisymmetric. For x and y that are equivalent (that is, both $x \preceq y$ and $y \preceq x$) the principal ideals $\downarrow x$ and $\downarrow y$ are equal. This means that the ideal completion construction automatically takes care of quotienting by the equivalence induced by $(\preceq)$.

Just as the structure of an algebraic cpo is completely determined by its basis, a continuous function from an algebraic cpo to another type is completely determined by its action on basis elements. This suggests a method for defining continuous functions over ideal completions: First, define a function from the basis B to a cpo C such that f is monotone, i.e. $x \preceq y$ implies $f(x) \sqsubseteq f(y)$. Then there exists a unique continuous function $\widehat{f} : \mathrm{Idl}(B) \to C$ that agrees with f on principal ideals, i.e. for all x, $\widehat{f}(\downarrow x) = f(x)$. We say that $\widehat{f}$ is the *continuous extension* of f.

In the next section, all of the constructions related to the universal domain will be done in terms of basis values: The universal domain itself will be defined using ideal completion, and the embedding and projection functions will be defined as continuous extensions.

HOLCF includes a formalization of ideal completion and continuous extensions, which was created to support the definition of powerdomains [8].

3 Construction of the Universal Domain

Informally, an *omega-bifinite domain* is a cpo that can be written as the limit of a sequence of finite partial orders. This section describes how to construct a *universal* omega-bifinite domain U, along with an ep-pair from another arbitrary omega-bifinite domain D into U. The general strategy is as follows:

– From the bifinite structure of D, obtain a sequence of finite posets P_n whose limit is D.
– Following Gunter [5], decompose the sequence P_n further into a sequence of *increments* that insert new elements one at a time.
– Construct the universal domain U using ideal completion, such that it can encode any increment.
– Define the embedding and projection functions between D and U using continuous extension, in terms of their action on basis elements.

3.1 Building a Sequence of Increments

Any omega-bifinite domain D can be represented as the limit of a sequence of finite posets, with embedding-projection pairs between each successive pair. Figure 2 shows the first few posets from one such sequence.

In each step along the chain, each new poset P_{n+1} is larger than the previous P_n by some finite amount; the structure of P_{n+1} has P_n embedded within it, but it has some new elements as well.

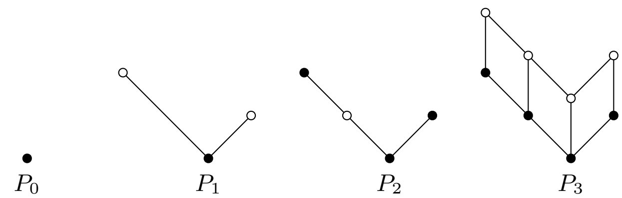


Fig. 2. A sequence of finite posets. Each P_n can be embedded into P_{n+1}; black nodes indicate the range of the embedding function.

An ep-pair between finite posets P and P', where P' has exactly one more element than P, is called an *increment* (terminology due to Gunter [6]). In Fig. 2, the embedding of P_1 into P_2 is an example of an increment.

The strategy for embedding a bifinite domain into the universal domain is built around increments. The universal domain is designed so that if a finite partial order P is representable (i.e. by a deflation), and there is an increment from P to P', then P' will also be representable.

For all embeddings from P_n to P_{n+1} that add more than one new value, we will need to decompose the single large embedding into a sequence of smaller increments. The challenge, then, is to determine which order the new elements should be inserted. The order matters: Adding elements in the wrong order can cause problems, as shown in Fig. 3.

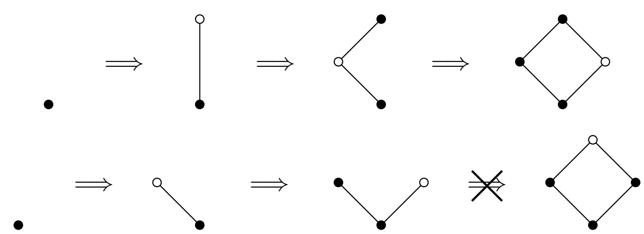


Fig. 3. The right (top) and wrong (bottom) way to order insertions. No ep-pair exists between the 3-element and 4-element posets on the bottom row.

To describe the position of a newly-inserted element, it will be helpful to invent some terminology. The set of elements *above* the new element will be known as its *superiors*. An element immediately *below* the new element will be known as its *subordinate*.

In order for the insertion of a new element to be a valid increment, it must have exactly one subordinate. The subordinate indicates the value that the increment's projection maps the new value onto.

With the four-element poset in the Fig. 3, it is not possible to insert the top element last. The reason is that the element has two subordinates: If a projection function maps the new element to one, the ordering relation with the other will not be preserved. Thus a monotone projection does not exist.

A strategy for successfully avoiding such situations is to always insert maximal elements first [5, §5]. Fig. 4 shows this strategy in action. Notice that the number of superiors varies from step to step, but each inserted element always has exactly one subordinate. To maintain this invariant, the least of the four new values must be inserted last.

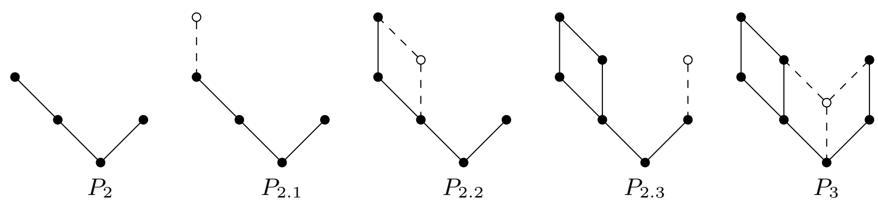


Fig. 4. A sequence of four increments going from P_2 to P_3. Each new node may have any number of upward edges, but only one downward edge.

Armed with this strategy, we can finally formalize the complete sequence of increments for type D. To each element x of the basis of D we must assign a sequence number $place(x)$—this numbering tells what order to insert the values. The HOLCF formalization breaks up the definition of *place* as follows. First, each basis value is assigned to a rank, where $rank(x) = n$ means that the basis value x first appears in the poset P_n. Equivalently, $rank(x)$ is the least n such that $approx_n(x) = x$. Then an auxiliary function assigns sequence numbers to values in finite sets, by repeatedly removing an arbitrary maximal element until the set is empty. Finally, $place(x)$ is defined as the sequence number of x within its (finite) rank set, plus the total size of all earlier ranks.

The full formalization of the *place* function is actually quite complex; many details are omitted here due to space limitations. Interested readers should refer to the HOLCF source. For the remainder of this paper, it will be sufficient to note that the *place* function satisfies the following two properties.

- Values in earlier ranks come before values in later ranks: If $rank(x) < rank(y)$, then $place(x) < place(y)$.
- Within the same rank, larger values come first: If $rank(x) = rank(y)$ and $x \sqsubseteq y$, then $place(y) < place(x)$.

3.2 A Basis for the Universal Domain

Constructing a partial order incrementally, there are two possibilities for any newly inserted value:

– The value is the very first one (i.e. it is $\bot$)
– The value is inserted above some previous value (its subordinate), and below zero or more other previous values (its superiors)

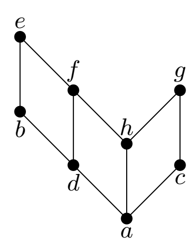
Accordingly, we can define a datatype to describle the position of these values relative to each other. (Usage of Haskell datatype syntax is merely for convenience; this is not intended to be viewed as a lazy datatype. Here `Nat` represents the natural numbers, and `Set a` represents finite sets with elements of type `a`.)

```
data Basis = Bottom | Node { serial_number :: Nat
                           , subordinate :: Basis
                           , superiors :: Set Basis }
```

The above definition does not work as a datatype definition in Isabelle/HOL, because the finite set type constructor does not work with the datatype package. (Indirect recursion only works with other inductive datatypes.) But it turns out that we do not need the datatype package at all—the type `Basis` is actually isomorphic to the natural numbers. Using the bijections $\mathbb{N} \cong 1+\mathbb{N}$ and $\mathbb{N} \cong \mathbb{N}\times\mathbb{N}$ with $\mathbb{N} \cong \mathcal{P}_f(\mathbb{N})$, we can construct a bijection that lets us use $\mathbb{N}$ as the basis datatype:

$$\mathbb{N} \cong 1 + \mathbb{N} \times \mathbb{N} \times \mathcal{P}_f(\mathbb{N}) \tag{2}$$

In the remainder of this section, we will use mathematical notation to write values of the basis datatype: $\bot$ represents `Bottom`, and $\langle i, a, S\rangle$ will stand for `Node i a s`.



$$\begin{aligned} a &= \bot \\ b &= \langle 1, a, \{\}\rangle \\ c &= \langle 2, a, \{\}\rangle \\ d &= \langle 3, a, \{b\}\rangle \\ e &= \langle 4, b, \{\}\rangle \\ f &= \langle 5, d, \{e\}\rangle \\ g &= \langle 6, c, \{\}\rangle \\ h &= \langle 7, a, \{e, f, g\}\rangle \end{aligned}$$

Fig. 5. Embedding elements of P_3 into the universal domain basis.

Figure 5 shows how this system works for embedding all the elements from the poset P_3 into the basis datatype. The elements have letter names from a–h, assigned alphabetically by insertion order. In the datatype encoding of each element, the subordinate and superiors are selected from the set of previously inserted elements. Serial numbers are assigned sequentially.

The serial number is necessary to distinguish multiple values that are inserted in the same position. For example, in Fig. 5, elements b and c both have a as the subordinate, and neither has any superiors. The serial number is the only way to tell such values apart.

Note that the basis datatype seems to contain some junk—some subordinate/superiors combinations are not well formed. For example, in any valid increment, all of the superiors are positioned above the subordinate. One way to take care of this requirement would be to define a well-formedness predicate for basis elements. However, it turns out that it is possible (and indeed easier) to simply ignore any invalid elements. In the set of superiors, only those values that are above the subordinate will be considered. (This will be important to keep in mind when we define the basis ordering relation.)

There is also a possibility of multiple representations for the same value. For example, in Fig. 5 the encoding of h is given as $\langle 7, a, \{e, f, g\}\rangle$, but the representation $\langle 7, a, \{f, g\}\rangle$ would work just as well (since the sets have the same upward closure). One could consider having a well-formedness requirement for the set of superiors to be upward-closed. But this turns out not to be necessary, since the extra values do not cause problems for any of the formal proofs.

3.3 Basis ordering relation

To perform the ideal completion, we need to define a preorder relation on the basis. The basis value $\langle i, a, S\rangle$ should fall above a and below all the values in set S that are above a. Accordingly, we define the relation $(\preceq)$ as the smallest reflexive, transitive relation that satisfies the following two introduction rules:

$$a \preceq \langle i, a, S\rangle \tag{3}$$

$$a \preceq b \land b \in S \Longrightarrow \langle i, a, S\rangle \preceq b \tag{4}$$

Note that the relation $(\preceq)$ is not antisymmetric. For example, we have both $a \preceq \langle i, a, \{a\}\rangle$ and $\langle i, a, \{a\}\rangle \preceq a$. However, for ideal completion this does not matter. Basis values a and $\langle i, a, \{a\}\rangle$ generate the same principal ideal, so they will be identified as elements of the universal domain.

Also note the extra hypothesis $a \preceq b$ in Eq. (4). Because we have not banished ill-formed subordinate/superiors combinations from the basis datatype, we must explicitly consider only those elements of the set of superiors that are above the subordinate.

3.4 Building the Embedding and Projection

In the HOLCF formalization, the embedding function *emb* from D to U is defined using continuous extension. The first step is to define *emb* on basis elements, generalizing the pattern shown in Fig. 5. The definition below uses well-founded recursion—all recursive calls to *emb* are on previously inserted values with smaller *place* numbers:

$$
\begin{aligned}
emb(x) &= \begin{cases} \bot & \text{if } x = \bot \\ \langle i, a, S \rangle & \text{otherwise} \end{cases} \\
\text{where } i &= place(x) \\
a &= emb(sub(x)) \\
S &= \{emb(y) \mid place(y) < place(x) \land x \sqsubseteq y\}
\end{aligned}
\tag{5}
$$

The subordinate value a is computed using a helper function *sub*, which is defined as $sub(x) = approx_{n-1}(x)$, where $n = rank(x)$. The ordering produced by the *place* function ensures that no previously inserted value with the same rank as x will be below x. Therefore the previously inserted value immediately below x must be $sub(x)$, which comes from the previous rank.

In order to complete the continuous extension, it is necessary to prove that the basis embedding function is monotone. That is, we must show that for any x and y in the basis of D, $x \sqsubseteq y$ implies $emb(x) \preceq emb(y)$. The proof is by well-founded induction over the maximum of $place(x)$ and $place(y)$. There are two main cases to consider:

- Case $place(x) < place(y)$: Since $x \sqsubseteq y$, it must be the case that $rank(x) < rank(y)$. Then, using the definition of *sub* it can be shown that $x \sqsubseteq sub(y)$; thus by the inductive hypothesis we have $emb(x) \preceq emb(sub(y))$. Also, from Eq. (3) we have $emb(sub(y)) \preceq emb(y)$. Finally, by transitivity we have $emb(x) \preceq emb(y)$.
- Case $place(y) < place(x)$: From the definition of *sub* we have $sub(x) \sqsubseteq x$. By transitivity with $x \sqsubseteq y$ this implies $sub(x) \sqsubseteq y$; therefore by the inductive hypothesis we have $emb(sub(x)) \preceq emb(y)$. Also, using Eq. (5), we have that $emb(y)$ is one of the superiors of $emb(x)$. Ultimately, from Eq. (4) we have $emb(x) \preceq emb(y)$.

The projection function *prj* from U to D is also defined using continuous extension. The action of *prj* on basis elements is specified by the following recursive definition:

$$
prj(a) = \begin{cases} emb^{-1}(a) & \text{if } \exists x.\ emb(x) = a \\ prj(subordinate(a)) & \text{otherwise} \end{cases}
\tag{6}
$$

To ensure that *prj* is well-defined, there are a couple of things to check. First of all, the recursion always terminates: In the worst case, repeatedly taking the subordinate of any starting value will eventually yield $\bot$, at which point the first branch will be taken since $emb(\bot) = \bot$. Secondly, note that emb^{-1} is uniquely defined, because *emb* is injective. Injectivity of *emb* is easy to prove, since each embedded value has a different serial number.

Just like with *emb*, we also need to prove that the basis projection function *prj* is monotone. That is, we must show that for any a and b in the basis of U, $a \preceq b$ implies $prj(a) \sqsubseteq prj(b)$. Remember that the basis preorder $(\preceq)$ is an inductively defined relation; accordingly, the proof proceeds by induction on $a \preceq b$. Compared to the proof of monotonicity for *emb*, the proof for *prj* is relatively straightforward; details are omitted here.

Finally, we must prove that *emb* and *prj* form an ep-pair. The proof of $prj \circ emb = \mathrm{Id}_D$ is easy: Let x be any value in the basis of D. Then using Eq. (6), we have $prj(emb(x)) = emb^{-1}(emb(x)) = x$. Finally, since the equation is an admissible predicate on x, this is sufficient to show that it holds for all values in the ideal completion.

The proof of $emb \circ prj \sqsubseteq \mathrm{Id}_U$ takes a bit more work. As a lemma, we can show that for any a in the basis of U, $prj(a)$ is always equal to $emb^{-1}(b)$ for some $b \preceq a$ that is in the range of *emb*. Using this lemma, we then have $emb(prj(a)) = emb(emb^{-1}(b)) = b \preceq a$. Finally, using admissibility, this is sufficient to show that $emb(prj(a)) \sqsubseteq a$ for all a in U.

To summarize the results of this section: We have formalized a type U, and two polymorphic continuous functions *emb* and *prj*. For any omega-bifinite domain D, *emb* and *prj* form an ep-pair that embeds D into U. The full proof scripts are available as part of recent developer versions of the Isabelle, in the theory file `src/HOLCF/Universal.thy`.

4 Integration with the Domain Package

There are several remaining steps involved in building a new domain package that uses the universal domain U.

1. Define a type T consisting of all the deflations over U whose image sets are omega-bifinite cpos. Each *value* of type T represents a *type*. Note that the type T is itself a cpo; this is important because it lets us use a fixed-point combinator to define recursive values of type T, representing recursive types.
2. For each of the basic type constructors in HOLCF, define a deflation combinator as a continuous function over type T. There will be combinators for cartesian product, continuous function space, strict sums and products, lifting, and basic types like *unit* and *bool*. Combinators for powerdomains could also be defined.

$$\begin{aligned} \times_T, \rightarrow_T, \oplus_T, \otimes_T &:: T \rightarrow T \rightarrow T \\ \mathrm{lift}_T &:: T \rightarrow T \\ \mathrm{unit}_T, \mathrm{bool}_T, \mathrm{int}_T &:: T \end{aligned}$$

3. Use the deflation combinators, together with a least fixed-point operator, to define deflations for recursive types. For example, the `BoolList` type used earlier should be a solution to the domain equation $D \cong \mathrm{unit}_T \oplus_T (\mathrm{bool}_T \otimes_T D)$. Accordingly, it should be defined using the least fixed-point combinator as $\mu D.\ \mathrm{unit}_T \oplus_T (\mathrm{bool}_T \otimes_T D)$.
4. Define each actual recursive type as the image set of its corresponding deflation. The isomorphism and induction rules (generated as axioms in the current implementation) will be derived from the fixed-point properties.

5. For each newly-defined recursive type (or type constructor), keep track of its deflation (or deflation combinator) for use in defining other datatypes.

Some of the above items have been formalized in previous work with Matthews and White [9], but not in the context of omega-bifinite cpos.

A complete implementation of the design described above would allow users to define many interesting datatypes that are not currently supported. Examples include uses of indirect recursion through previously-defined datatypes, and also through type constructors defined outside the domain package, like powerdomains. Other examples include datatypes with negative recursion, like the continuation-based `Resumption` datatype from the introduction.

This brings us back to the questions about the `Resumption` type posed in the introduction. The induction rule in Eq. (1) can be derived from the fixed-point induction rule for the deflation used to model type `Resumption r a`. Due to the indirect recursion, the deflation for type `Resumption r a` is defined in terms of the deflation combinator for type `Cont r`. Furthermore, the deflation combinator happens to coincide with the function `mapCont`, which explains its appearance in Eq. (1). (This is similar to how the function `map` doubles as the deflation combinator for lists. In general, a map function that satisfies the functor laws will coincide with the deflation combinator for that type.)

5 Related Work

An early example of the purely definitional approach to defining datatypes is described by Melham, in the context of the HOL theorem prover [10]. Melham defines a type $(\alpha)\,\mathit{Tree}$ of labelled trees, from which other recursive types are defined as subsets. The design is similar in spirit to the one presented in this paper—types are modeled as values, and abstract axioms that characterize each datatype are proved as theorems. The main differences are that it uses ordinary types instead of bifinite domains, and ordinary subsets instead of deflations.

The Isabelle/HOL datatype package uses a design very similar to the HOL system. The type α *node*, which was originally used for defining recursive types in Isabelle/HOL, was introduced by Paulson [13]; it is quite similar to the HOL system's $(\alpha)\,\mathit{Tree}$ type. Gunter later extended the labelled tree type of HOL to support datatypes with arbitrary branching [7]. Berghofer and Wenzel used a similarly extended type to implement Isabelle's modern datatype package [2].

The Coq theorem prover takes an alternative approach to defining datatypes. Unlike Isabelle or the HOL system, the semantics of datatypes in Coq are not formalized within the prover itself. The rules and limitations for inductive datatype definitions in Coq are determined outside the logic, as described by Paulin-Mohring [12]. However, compared to Isabelle or HOL, Coq supports many more of the kinds of recursive datatypes found in Haskell, including higher-order and non-regular datatypes. The semantics of such datatypes in Coq will be relevant if the domain package is ever extended to support them.

On the domain theory side, various publications by Gunter [4–6] were the primary sources of ideas for my universal domain construction. The construction of the sequence of increments in Section 3 is just as described by Gunter [5, §5]. However, the use of ideal completion is original—Gunter defines the universal domain using a colimit construction instead. Given a cpo D, Gunter defines a type D^+ that can embed any increment from D to D'. The universal domain is then defined as a solution to the domain equation $D = D^+$. The construction of D^+ is similar to my `Basis` datatype, except that it is non-recursive and does not include serial numbers.

Amadio and Curien [1] describe how to construct a model for Type:Type using a universal bounded-complete domain. The same method should work for omega-bifinite domains as well: Since the type T of deflations over U is itself omega-bifinite, type T contains a value that represents type T. This offers the possibility of modeling higher-rank polymorphism in HOLCF.

6 Conclusion and Future Work

The Isabelle/HOLCF library of domain theory now provides a universal domain type, into which any omega-bifinite domain can be embedded. The set of deflations over the universal domain includes representations for a large class of recursive datatypes; thus it forms a suitable starting point for building a purely definitional datatype package for HOLCF.

Planned future work consists primarily of implementing the domain package design outlined in Section 4. Other possible areas for future work involve exploring limitations in the current design:

– *Higher-order type constructors.* Higher-order types can be represented by deflation combinators with types like $(T \to T) \to T$. The problem is that Isabelle's type system only supports first-order types. Although, see [9] for an admittedly complicated workaround.
– *Non-regular (nested) datatypes* [3]. Deflation combinators for non-regular datatypes can be defined by taking least fixed points at type $T \to T$, rather than type T. Isabelle's type system can also handle the types of the constructors. The problem is that since Isabelle does not support type quantification or polymorphic recursion, induction rules and recursive functions could not be defined in the normal way. However, even if the normal automation is not available, it might make sense to support such datatypes in HOLCF; users could define and prove things manually by unfolding the definitions.
– *Higher-rank polymorphism.* This is not supported by Isabelle's type system. However, the universal domain U could be used to model such types, using the construction described by Amadio and Curien [1].
– *Generalized abstract datatypes (GADTs).* These are usually modeled in terms of some kind of type equality constraints. For example, type equality constraints are a central feature of System F_C [15], which models the internal representation of Haskell programs used by the Glasgow Haskell Compiler (GHC). But to the extent of this author's knowledge, there is no way to model type equality constraints using deflations.

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