Extending Hindley-Milner Type Inference with Coercive Structural Subtyping

Dmitriy Traytel Stefan Berghofer Tobias Nipkow

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Technische Universität München

Outline

Why coercions?

A naive algorithm

Constraint-based algorithm

Conclusion

Real-world examples

- 2004: Avigad verifies in Isabelle:

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- 2009: Hölzl uses 1061 explicit conversions in a single theory
- Both report “headaches”

Solution: coercive structural subtyping

Related work

- Subtyping part of the type system:

  Mitchell, Fuh & Mishra, Wand & O'Keefe, Pottier, Simonet

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- Complete coercion inference system:

  this publication

The Hindley-Milner typing rules remain unchanged:
No subtypes here

Type inference is extended with coercion inference
and coercion insertion

Our coercion inference system

- Coercions: $\mathbb{N} <:_{\texttt{real}} \mathbb{R}$
- Lifted by map functions: $\mathbb{N}$ *list* $<:_{\texttt{map real}} \mathbb{R}$ *list*
- Programmer inputs terms omitting coercions
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- Programmer inputs terms omitting coercions
- The system infers and inserts coercions
- Result is well-typed according to Hindley-Milner
- The coercion inference system:
  - is sound and complete
  - does not change the underlying type system

Local coercion insertion

- Use judgement $\Gamma \vdash \mathtt{t} \leadsto \mathtt{u} : \tau$
- Idea: insert coercions whenever the function's domain does not match the argument type:

$$\frac{\vdash \mathtt{t}_1 \leadsto \mathtt{u}_1 : \tau_{11} \rightarrow \tau_{12} \qquad \vdash \mathtt{t}_2 \leadsto \mathtt{u}_2 : \tau_2 \qquad \tau_2 <:_{\mathtt{c}} \tau_{11}}{\vdash \mathtt{t}_1\ \mathtt{t}_2 \leadsto \mathtt{u}_1\ (\mathtt{c}\ \mathtt{u}_2) : \tau_{12}}$$

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- Used in Coq

Problematic example

Example: `leq i n` vs. `leq n i`

- Signatures: $\texttt{leq} :: \alpha \to \alpha \to \mathbb{B}$, $\texttt{n} :: \mathbb{N}$ and $\texttt{i} :: \mathbb{Z}$
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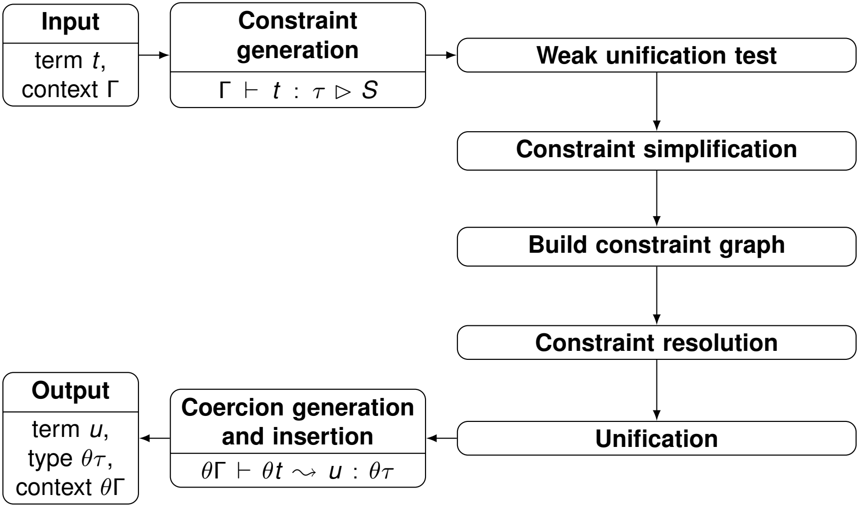
- Correctly, `leq i n` becomes `leq i (int n)`, as
  - $\texttt{leq i} :: \mathbb{Z} \to \mathbb{B}$
  - $\texttt{n} :: \mathbb{N}$
- Unfortunately, the coercion inference of `leq n i` fails, as
  - $\texttt{leq n} :: \mathbb{N} \to \mathbb{B}$
  - $\texttt{i} :: \mathbb{Z}$
  - no coercion from $\mathbb{Z}$ to $\mathbb{N}$

This is “normal” behaviour of coercions.

Coq Reference Manual

The subtyping pipeline



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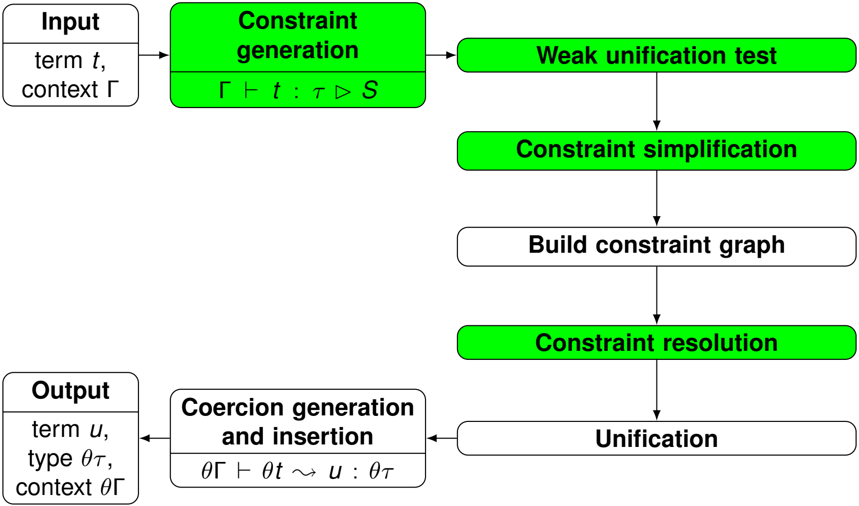

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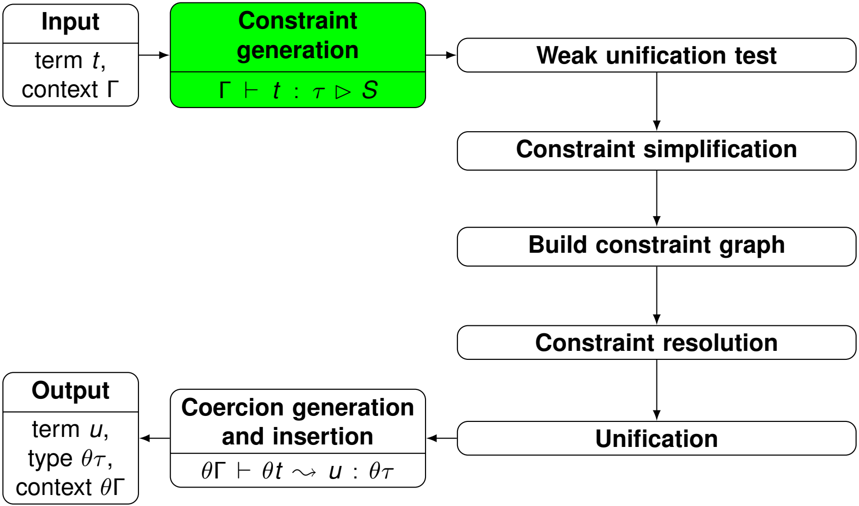

Constraint generation

$$\frac{\vdash \texttt{t}_1 : \tau \rhd S_1 \qquad \vdash \texttt{t}_2 : \sigma \rhd S_2 \qquad \alpha, \beta \text{ fresh}}{\vdash \texttt{t}_1\ \ \texttt{t}_2 : \beta \rhd S_1 \cup S_2 \cup \{\tau \doteq \alpha \to \beta, \sigma <: \alpha\}}$$

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Example: `leq n i`

$$\dfrac{\dfrac{\dfrac{\texttt{leq} :: \alpha \to \alpha \to \mathbb{B}}{\vdash \texttt{leq} : \alpha \to \alpha \to \mathbb{B} \rhd \emptyset} \qquad \dfrac{\texttt{n} :: \mathbb{N}}{\vdash \texttt{n} : \mathbb{N} \rhd \emptyset}}{\vdash \texttt{leq}\ \ \texttt{n} : \beta_2 \rhd \{\alpha \to \alpha \to \mathbb{B} \doteq \alpha_2 \to \beta_2,\ \mathbb{N} <: \alpha_2\}} \qquad \dfrac{\texttt{i} :: \mathbb{Z}}{\vdash \texttt{i} : \mathbb{Z} \rhd \emptyset}}{\vdash \texttt{leq}\ \ \texttt{n}\ \ \texttt{i} : \beta_1 \rhd \left\{ \begin{array}{rcl} \alpha \to \alpha \to \mathbb{B} & \doteq & \alpha_2 \to \beta_2, \\ \beta_2 & \doteq & \alpha_1 \to \beta_1, \\ \mathbb{N} & <: & \alpha_2, \\ \mathbb{Z} & <: & \alpha_1 \end{array} \right\}}$$

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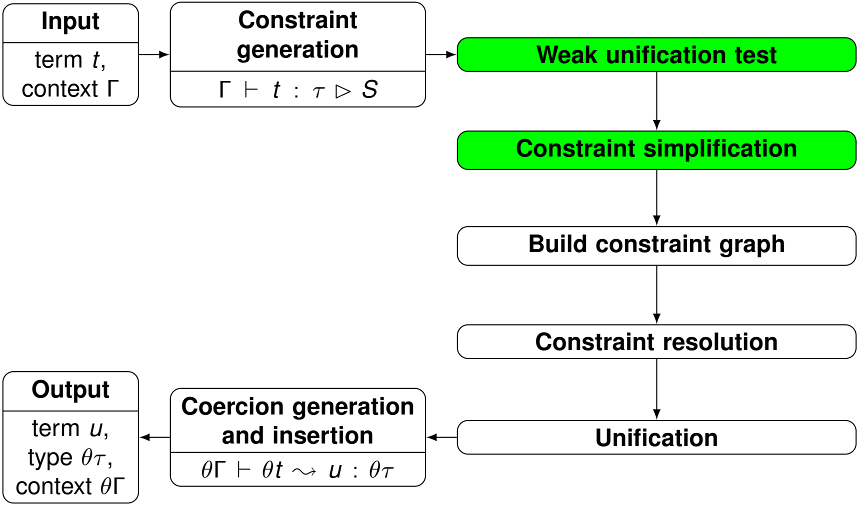

Constraint simplification

- Goal: only atomic constraints $\alpha <: \beta$, $\alpha <: T$, $T <: \alpha$

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$$\sigma_1 \to \sigma_2 <: \tau_1 \to \tau_2 \quad \Leftrightarrow \quad \tau_1 <: \sigma_1 \text{ and } \sigma_2 <: \tau_2$$

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- $\Rightarrow$ corresponds to simplification
- $\Leftarrow$ corresponds to coercion generation
- variances are derived from map functions
  - `map` $:: (\alpha \rightarrow \beta) \rightarrow \alpha \ \mathit{list} \rightarrow \beta \ \mathit{list}$
  - λ`f g h. g ∘ h ∘ f` $::$ $(\beta_1 \rightarrow \alpha_1) \rightarrow (\alpha_2 \rightarrow \beta_2) \rightarrow (\alpha_1 \rightarrow \alpha_2) \rightarrow (\beta_1 \rightarrow \beta_2)$

Weak unification

$$\alpha <: \alpha \textit{ list} \quad \Leftrightarrow \quad \exists \alpha'.\, \alpha \doteq \alpha' \textit{ list} \text{ and } \alpha' \textit{ list} <: \alpha \textit{ list}$$

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- Not solvable with structural coercions

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- Simplification process does not terminate
- Not solvable with structural coercions
- Weak unification := unification after identifying all base types
- Initial constraint set weakly unifiable $\Rightarrow$ termination proof

Constraint simplification (example)

Example: `leq n i`

$$\{\alpha \to \alpha \to \mathbb{B} \doteq \alpha_2 \to \beta_2,\ \beta_2 \doteq \alpha_1 \to \beta_1,\ \mathbb{N} <: \alpha_2,\ \mathbb{Z} <: \alpha_1\}$$

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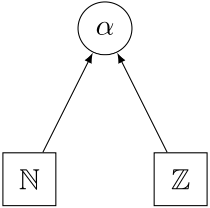

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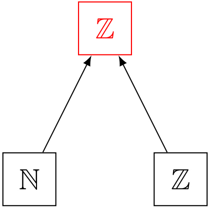

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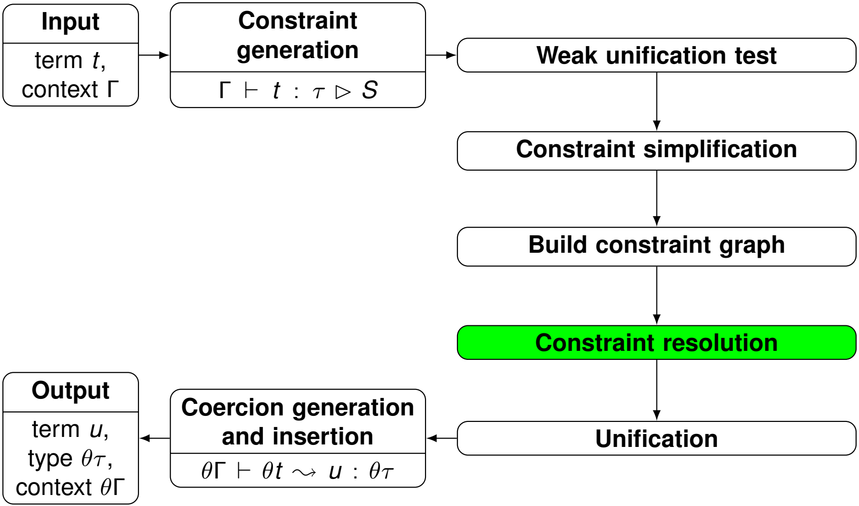
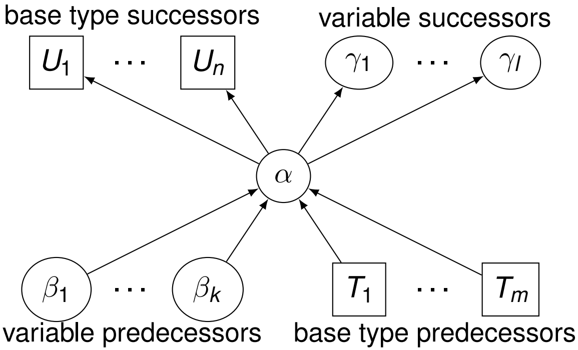
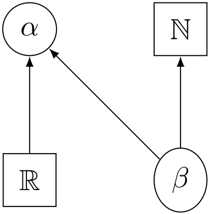


Constraint resolution



- Compute the intersection of sets of all supertypes of base type predecessors of α
- Assign α the "smallest" type from the intersection
- Check that the assignment is subtype of all base type successors

Constraint resolution (example)



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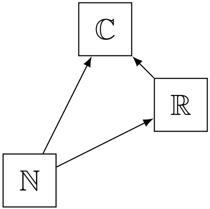

Partial order on base types

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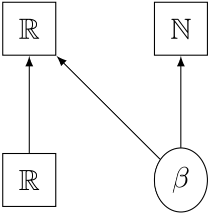

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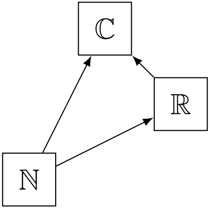

Partial order on base types

- Possibly, the algorithm assigns α the type $\mathbb{R}$ first

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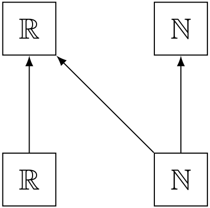

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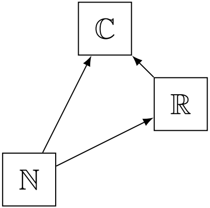

Partial order on base types

- Possibly, the algorithm assigns α the type $\mathbb{R}$ first
- Then β is assigned the infimum of $\{\mathbb{N}, \mathbb{R}\}$

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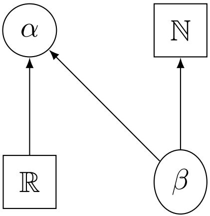

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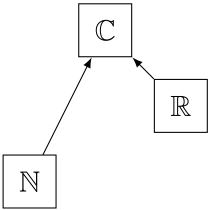

Partial order on base types

- Same constraints, different coercion declarations

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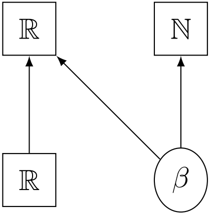

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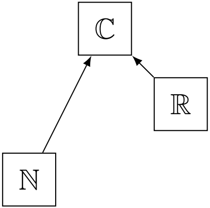

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$\Rightarrow$ Coercion inference fails

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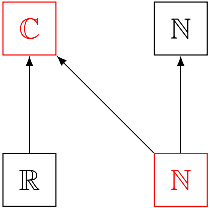

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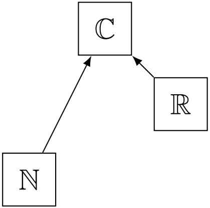

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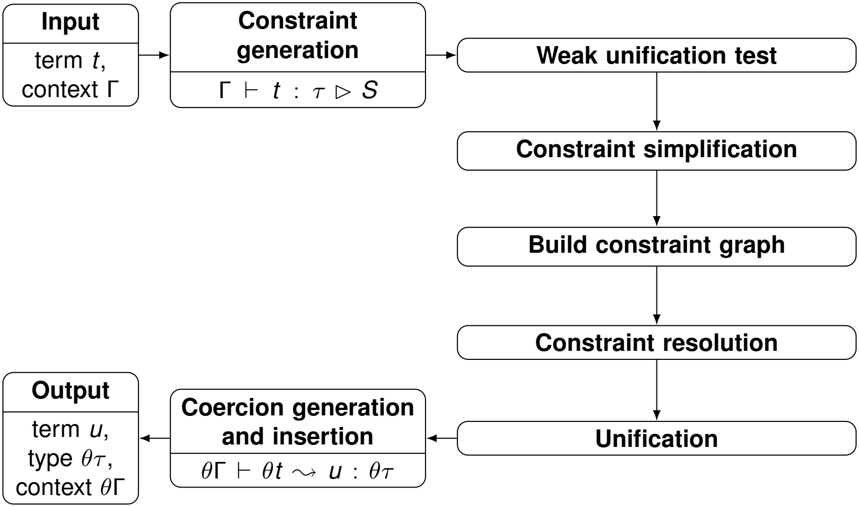
- But: $\{\alpha \mapsto \mathbb{C},\ \beta \mapsto \mathbb{N}\}$ is a solution

The subtyping pipeline



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  - Assumption: subtyping relation is a disjoint union of lattices
  - If $\mathtt{t}$ can be coerced to a well-typed term $\mathtt{u}$ in the context Γ, then the algorithm will output a term $\mathtt{u'}$
  - Can't guarantee $\mathtt{u} = \mathtt{u'}$
  - $\Rightarrow$ refined notion of completeness

Ambiguity example

Example: `sin (- n)`

- Signatures: $\texttt{sin} :: \mathbb{R} \to \mathbb{R}$, $\texttt{-} :: \alpha \to \alpha$ and $\texttt{n} :: \mathbb{N}$
- Declared coercion: $\mathbb{N} <:_{\texttt{real}} \mathbb{R}$

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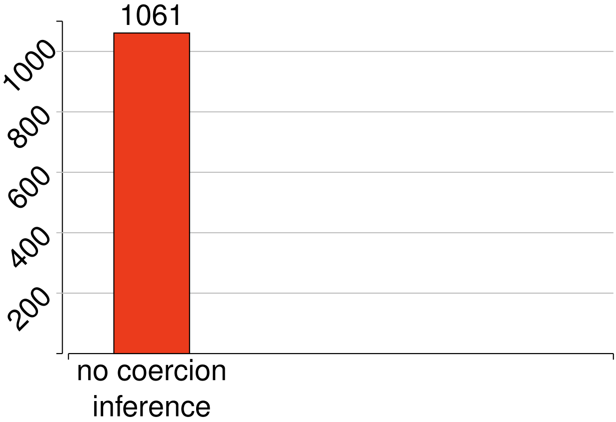
- Declared coercion: $\mathbb{N} <:_{\texttt{real}} \mathbb{R}$
- Two possible output terms:
  - `sin (real (- n))`
  - `sin (- (real n))`

Headache reduction factor

- Necessary coercions in Hölzl's theory



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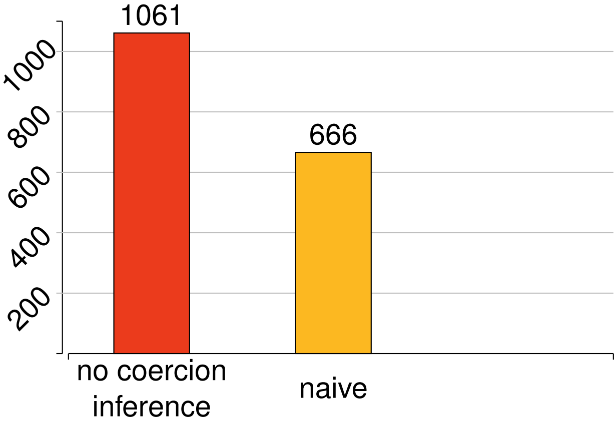

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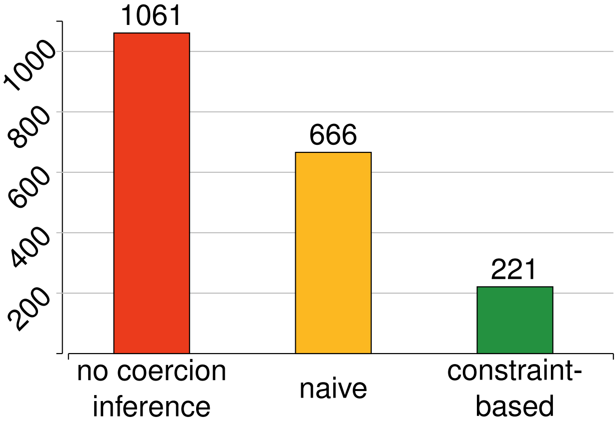

Thank you for your attention!

Questions?

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- Declared coercions: $\mathbb{N} <:_{\texttt{int}} \mathbb{Z}$, $\mathbb{Z} <:_{\texttt{real}} \mathbb{R}$
- Derived coercion: $\mathbb{N} <:_{\texttt{real} \circ \texttt{int}} \mathbb{R}$

Another ambiguity example

Example: `sin (- n)`

- Signatures: $\texttt{sin} :: \mathbb{R} \to \mathbb{R}$, $- :: \alpha \to \alpha$ and $\texttt{n} :: \mathbb{N}$
- Declared coercions: $\mathbb{N} <:_{\texttt{int}} \mathbb{Z}$, $\mathbb{Z} <:_{\texttt{real}} \mathbb{R}$
- Derived coercion: $\mathbb{N} <:_{\texttt{real} \circ \texttt{int}} \mathbb{R}$
- Two possible output terms:
  - `sin ((real ∘ int) (- n)))`
  - `sin (- ((real ∘ int) n)))`
- Impossible output term:
  - `sin (real (- (int n)))`

Coercive subtyping and *let*-polymorphism

Example: `let f = s in u`
where $\mathtt{s} \equiv \lambda\mathtt{x.\ if\ x > n} \wedge \mathtt{sin\ x > r\ then\ x\ else\ x}$
and $\mathtt{u} \equiv \mathtt{(Suc\ (f\ n),\ f\ r)}$

- Signatures: $\Sigma(\mathtt{sin}) = \mathbb{R} \to \mathbb{R}$, $\Sigma(\mathtt{Suc}) = \mathbb{N} \to \mathbb{N}$, $\Sigma(>) = \alpha \to \alpha \to \mathbb{B}$, $\Sigma(\mathtt{n}) = \mathbb{N}$ and $\Sigma(\mathtt{r}) = \mathbb{R}$
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Coercive subtyping and *let*-polymorphism

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- `let f = s in u` is not coercible either way
- On the other hand `u[s/f]` can be coerced