Proof of Herr Schmidmeiers' Solution

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Lemma 1. *Let $G = (V, E)$ be a fully connected directed graph, let CO and TC be defined as in H2.2. (with changed edge directions) and suppose $TC \neq CO$. Then there exists at least one cycle of length 3 and at least one cycle of length 4 in TC.*

Proof. As proved in my solution, $CO \subsetneq TC$ holds in this case.
Further

$$\exists u \in CO : \exists v \in N_{\text{out}}(u) \setminus CO$$

as otherwise $TC = CO$ would hold. Let $k := \deg_{\text{out}}(u)$.
We know by definition of CO that $deg_{\text{out}}(v) \geq k + 1$ and due to $v \notin N_{\text{out}}(u)$ we get

$$|N_{\text{out}}(v) \setminus N_{\text{out}}(u)| \geq 2$$

Now let $w, w' \in N_{\text{out}}(v) \setminus N_{\text{out}}(u), w \neq w'$. W.l.o.g. assume $(w, w') \in E$. In this case

$$C_3 := (u, v, w)$$
$$C_4 := (u, v, w, w')$$

are cycles of length 3 and 4 respectively, as the graph is fully connected (i.e. $w \notin N_{\text{out}}(u) \Leftrightarrow u \in N_{\text{out}}(w)$). □

Lemma 2. *Following holds:*

$$\forall n \geq 6 \exists k, l \in \mathbb{N} : n = 3k + 4l$$

Proof. Let $n = 3m + r$ where $r \in \{0, 1, 2\}$. (This is a basic property of natural numbers). Due to $n \geq 6$ we get $m \geq 2$.
Setting $k = (m - r), l = r$ does the job:

$$n = 3m + r = 3(m - r) + 3 \cdot r + r = 3 \cdot (m - r) + 4 \cdot r$$

□

Corollary 1. *The solution proposed by Herr Schmidmeier does terminate.*

Proof. We define G by changing the direction of all edges in the tournament graph which can be found in Figure 1 on the exercise sheet. Hence, if we use the notation from the Wettbewerb Website, $\bar{D}(v) = N_{\text{out}}(v)$. Let $n := |TC|$. Let $u \in V$ be the vertex defined in the proof of Lemma 1. As proven in my solution, the algorithm does also terminate if started from only u, hence there must be a path (of maximum length $n-1$) from u to every vertex in TC. Also, if starting from u, by Lemma 1. and 2., u will be in every iteration after the 6^{th}. Hence, after a maximum of $n+5$ iterations, Herr Schmidmeier will find the correct TC. □