Informatik 2:
Functional Programming

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Sprungtabelle

▸ 16. Oktober ▸ 23. Oktober ▸ 30. Oktober

▸ 6. November ▸ 13. November ▸ 20. November ▸ 27. November

▸ 4. Dezember ▸ 11. Dezember ▸ 18. Dezember

▸ 8. Januar ▸ 15. Januar ▸ 22. Januar ▸ 29. Januar

1. Organisatorisches
2. Functional Programming: The Idea
3. Basic Haskell
4. Lists
5. Proofs
6. Higher-Order Functions
7. Type Classes
8. Algebraic `data` Types
9. Modules and Abstract Data Types
10. Case Study: Huffman Coding
11. Case Study: Parsing
12. Lazy evaluation
13. I/O and Monads
14. Complexity and Optimization

1. Organisatorisches

Siehe `http://fp.in.tum.de`

Wochenplan

Dienstag	Vorlesung: Gehirn mitbringen
	Harter Abgabetermin für Übungsblatt
	Neues Übungsblatt
Mi–Fr	Übungen: Gehirn *und* Laptop mitbringen

Literatur

- Vorlesung orientiert sich stark an
  Thompson: *Haskell, the Craft of Functional Programming*
- Für Freunde der kompakten Darstellung:
  Hutton: *Programming in Haskell*
- Für Naturtalente: Es gibt sehr viel Literatur online.
  Qualität wechselhaft, nicht mit Vorlesung abgestimmt.

Klausur und Hausaufgaben

- Klausur am Ende der Vorlesung
- Wer mindestens 40% der Hausaufgabenpunkte erreicht und die Klausur besteht, bekommt einen Notenbonus von 0.3 (bei bestandener Klausur).
- Wer Hausaufgaben abschreibt oder abschreiben lässt, hat seinen Notenbonus sofort verwirkt.

2. Functional Programming: The Idea

Functions are pure/mathematical functions:
Always same output for same input

Computation = Application of functions to arguments

Example 1

In Haskell:

```
sum [1..10]
```

In Java:

```
total = 0;
for (i = 1; i <= 10; ++i)
  total = total + i;
```

Example 2

In Haskell:

```
wellknown [] = []
wellknown (x:xs) = wellknown ys ++ [x] ++ wellknown zs
   where ys = [y | y <- xs, y <= x]
         zs = [z | z <- xs, x < z]
```

In Java:

```
void sort(int[] values) {
   if (values ==null || values.length==0){ return; }
   this.numbers = values;
   number = values.length;
   quicksort(0, number - 1);
}

void quicksort(int low, int high) {
   int i = low, j = high;
   int pivot = numbers[low + (high-low)/2];
   while (i <= j) {
      while (numbers[i] < pivot) { i++; }
      while (numbers[j] > pivot) { j--; }
      if (i <= j) {exchange(i, j); i++; j--; }
   }
   if (low < j) quicksort(low, j);
   if (i < high) quicksort(i, high);
}

void exchange(int i, int j) {
   int temp = numbers[i];
   numbers[i] = numbers[j];
   numbers[j] = temp;
}
```

> *There are two ways of constructing a software design:*
>
> *One way is to make it so simple that there are obviously no deficiencies.*
>
> *The other way is to make it so complicated that there are no obvious deficiencies.*

From the Turing Award lecture by Tony Hoare (1985)

Characteristics of functional programs

elegant	
expressive	
concise	
readable	
predictable	pure functions, no side effects
provable	it's just (very basic) mathematics!

Aims of functional programming

- Program at a high level of abstraction:
  not bits, bytes and pointers but whole data structures
- Minimize time to read and write programs:
  ⇒ reduced development and maintenance time and costs
- Increased confidence in correctness of programs:
  clean and simple syntax and semantics
  ⇒ programs are easier to
    - understand
    - test (Quickcheck!)
    - prove correct

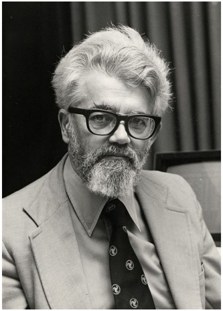
Historic Milestones

1930s

Alonzo Church develops the lambda calculus,
the core of all functional programming languages.

Historic Milestones

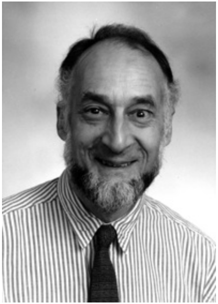
1950s



John McCarthy (Turing Award 1971) develops Lisp,
the first functional programming language.

Historic Milestones

1970s



Robin Milner (FRS, Turing Award 1991) & Co. develop ML, the first modern functional programming language with *polymorphic types* and *type inference*.

Historic Milestones

1987

An international committee of researchers initiates the development of Haskell, a standard lazy functional language.

Popular languages based on FP

F# (Microsoft) = ML for the masses

Erlang (Ericsson) = distributed functional programming

Scala (EPFL) = Java + FP

FP concepts in other languages

Garbage collection: Java, C#, Python, Perl, Ruby, Javascript

Higher-order functions: Java, C#, Python, Perl, Ruby, Javascript

Generics: Java, C#

List comprehensions: C#, Python, Perl 6, Javascript

Type classes: C++ “concepts”

Why we teach FP

- FP is a fundamental programming style (like OO!)
- FP is everywhere: Javascript, Scala, Erlang, F# ...
- It gives you the edge over Millions of Java/C/C++ programmers out there
- FP concepts make you a better programmer, no matter which language you use
- To show you that programming need not be a black art with magic incantations like `public static void` but can be a science

3. Basic Haskell

- Notational conventions
- Type `Bool`
- Type `Integer`
- Guarded equations
- Recursion
- Syntax matters
- Types `Char` and `String`
- Tuple types
- Do's and Don'ts

3.1 Notational conventions

`e::`T means that expression e has type T

Function types:	Mathematics	Haskell
	$f :: A \times B \to C$	`f :: A -> B -> C`

Function application:	Mathematics	Haskell
	$f(a)$	`f a`
	$f(a,b)$	`f a b`
	$f(g(b))$	`f (g b)`
	$f(a,g(b))$	`f a (g b)`

Prefix binds stronger than infix:

`f a + b` means `(f a) + b`
not `f (a + b)`

3.2 Type `Bool`

Predefined: `True False not && || ==`

Defining new functions:

```
xor :: Bool -> Bool -> Bool
xor x y  =  (x || y) && not(x && y)

xor2 :: Bool -> Bool -> Bool
xor2 True  True   =  False
xor2 True  False  =  True
xor2 False True   =  True
xor2 False False  =  False
```

This is an example of pattern matching.
The equations are tried in order. More later.

Is `xor x y == xor2 x y` true?

Testing with QuickQueck

Import test framework:

```
import Test.QuickCheck
```

Define property to be tested:

```
prop_xor2 x y =
  xor x y == xor2 x y
```

Note naming convention `prop_...`

Check property with GHCi:

```
> quickCheck prop_xor2
```

GHCi answers

```
+++ OK, passed 100 tests.
```

`BoolDemo.hs`

For GHCi commands (`:l` etc) see home page

3.3 Type `Integer`

Unlimited precision mathematical integers!

Predefined: `+ - * ^ div mod abs == /= < <= > >=`

There is also the type `Int` of 32-bit integers.

Warning:
```
Integer:  2 ^ 32 = 4294967296
Int:      2 ^ 32 = 0
```

==, <= etc are overloaded and work on many types!

Example:

```
sq :: Integer -> Integer
sq n  =  n * n
```

Evaluation:

$$\begin{aligned}\underline{\texttt{sq}}\ \texttt{(sq 3)} &= \underline{\texttt{sq}}\ \texttt{3 * }\underline{\texttt{sq}}\ \texttt{3}\\ &= \texttt{(3 * 3) * (3 * 3)}\\ &= \texttt{81}\end{aligned}$$

Evaluation of Haskell expressions
means
Using the defining equations from left to right.

3.4 Guarded equations

Example: maximum of 2 integers.

```
max :: Integer -> Integer -> Integer
max x y
  | x >= y      = x
  | otherwise   = y
```

Haskell also has `if-then-else`:

```
max x y  =  if x >= y then x else y
```

True?

```
prop_max_assoc x y z =
  max x (max y z) == max (max x y) z
```

3.5 Recursion

Example: x^n (using only $*$, not ^)

```
-- pow x n returns x to the power of n
pow :: Integer -> Integer -> Integer
pow x n  =  ???
```

Cannot write $\underbrace{x * \cdots * x}_{n \text{ times}}$

Two cases:

```
pow x n
  | n == 0  = 1                  -- the base case
  | n > 0   = x * pow x (n-1)    -- the recursive case
```

More compactly:

```
pow x 0  =  1
pow x n | n > 0  =  x * pow x (n-1)
```

Evaluating `pow`

```
pow x 0  =  1
pow x n | n > 0  =  x * pow x (n-1)
```

```
pow 2 3 = 2 * pow 2 2
        = 2 * (2 * pow 2 1)
        = 2 * (2 * (2 * pow 2 0))
        = 2 * (2 * (2 * 1))
        = 8
```

```
> pow 2 (-1)
```

GHCi answers

```
*** Exception: PowDemo.hs:(1,1)-(2,33):
    Non-exhaustive patterns in function pow
```

Partially defined functions

```
pow x n | n > 0  =  x * pow x (n-1)
```

versus

```
pow x n  =  x * pow x (n-1)
```

- call outside intended domain raises exception
- call outside intended domain leads to arbitrary behaviour, including nontermination

In either case:

State your preconditions clearly!

As a guard, a comment or using QuickCheck:

```
P x ==> isDefined(f x)
```

where `isDefined y  =  y == y`.

Example `sumTo`

The sum from 0 to `n` = `n + (n-1) + (n-2) + ...  + 0`

```
sumTo :: Integer -> Integer
sumTo 0  =  0
sumTo n | n > 0  =
```

```
prop_sumTo n =
  n >= 0 ==> sumTo n == n*(n+1) `div` 2
```

Properties can be *conditional*

Typical recursion patterns for integers

```
f :: Integer -> ...
f 0  =  e                                   -- base case
f n | n > 0  =  ... f(n - 1) ...    -- recursive call(s)
```

Always make the base case as simple as possible,
typically 0, not 1

Many variations:

- more parameters
- other base cases, e.g. `f 1`
- other recursive calls, e.g. `f(n - 2)`
- also for negative numbers

Recursion in general

- Reduce a problem to a *smaller* problem,
  e.g. `pow x n` to `pow x (n-1)`
- Must eventually reach a *base case*
- Build up solutions from smaller solutions

General problem solving strategy
in *any* programing language

3.6 Syntax matters

Functions are defined by one or more equations.
In the simplest case, each function is defined
by one (possibly conditional) equation:

$$
\begin{array}{l}
f\ \ x_1\ \ \ldots\ \ x_n \\
\quad \begin{array}{llll}
| & test_1 & = & e_1 \\
\vdots & & & \\
| & test_n & = & e_n
\end{array}
\end{array}
$$

Each right-hand side e_i is an expression.
Note: `otherwise = True`

Function and parameter names must begin with a lower-case letter
(Type names begin with an upper-case letter)

An *expression* can be

- a *literal* like `0` or `"xyz"`,
- or an *identifier* like `True` or `x`,
- or a *function application* $f\ e_1\ \ldots\ e_n$ where f is a function and $e_1\ \ldots\ e_n$ are expressions,
- or a parenthesized expression `(`e`)`

Additional syntactic sugar:

- `if then else`
- infix
- `where`
- `...`

Local definitions: `where`

A defining equation can be followed by one or more local definitions.

```
pow4 x  =  x2 * x2 where x2 = x * x

pow4 x  =  sq (sq x) where sq x = x * x

pow8 x  =  sq (sq x2)
  where x2 = x * x
        sq x = x * x

myAbs x
  | x > 0       =  y
  | otherwise   = -y
  where y = x
```

Local definitions: `let`

`let` x `=` e_1 `in` e_2

defines x locally in e_2

Example:

```
let x = 2+3 in x^2 + 2*x
= 35
```

Like e_2 `where` x `=` e_1
But can occur anywhere in an expression
`where`: only after function definitions

Layout: the offside rule

```
a = 10
b = 20
c = 30
```

~~`a = 10`~~
~~` b = 20`~~
~~`c = 30`~~

~~` a = 10`~~
~~`b = 20`~~
~~` c = 30`~~

In a sequence of definitions,
each definition must begin in the same column.

```
a = 10 +
  20
```

~~`a = 10 +`~~
~~`20`~~

~~` a = 10 +`~~
~~`20`~~

A definition ends with the first piece of text
in or to the left of the start column.

Prefix and infix

Function application: `f a b`

Functions can be turned into infix operators
by enclosing them in back quotes.

Example

```
5 ‘mod‘ 3  =  mod 5 3
```

Infix operators: `a + b`

Infix operators can be turned into functions
by enclosing them in parentheses.

Example

```
(+) 1 2  =  1 + 2
```

Comments

Until the end of the line: `--`

```
id x = x    -- the identity function
```

A comment block: `{- ... -}`

```
{- Comments
   are
   important
-}
```

3.7 Types `Char` and `String`

Character literals as usual: `'a'`, `'$'`, `'\n'`, ...
Lots of predefined functions in module `Data.Char`

String literals as usual: `"I am a string"`
Strings are lists of characters.
Lists can be concatenated with ++:
`"I am" ++ "a string"` = `"I ama string"`
More on lists later.

3.8 Tuple types

`(True, 'a', "abc") :: (Bool, Char, String)`

In general:

$$\begin{array}{ll} \text{If} & e_1 \texttt{ :: } T_1 \quad \ldots \quad e_n \texttt{ :: } T_n \\ \text{then} & \texttt{(}e_1\texttt{,}\ldots\texttt{,}e_n\texttt{) :: (}T_1\texttt{,}\ldots\texttt{,}T_n\texttt{)} \end{array}$$

In mathematics: $T_1 \times \ldots \times T_n$

3.9 Do's and Don'ts

`True and False`

Never write

$$b \texttt{ == True}$$

Simply write

$$b$$

Never write

$$b \texttt{ == False}$$

Simply write

$$not(b)$$

```
isBig :: Integer -> Bool
```

~~isBig n~~
~~ | n > 9999   = True~~
~~ | otherwise = False~~

```
isBig n  =  n > 9999
```

~~if b then True else False~~	b
~~if b then False else True~~	not b
~~if b then True else b’~~	b || b’

...

Tuple

Try to avoid (mostly):

```
f (x,y)  =  ...
```

Usually better:

```
f x y  =  ...
```

Just fine:

```
f x y  =  (x + y, x - y)
```

4. Lists

List comprehension
Generic functions: Polymorphism
Case study: Pictures
Pattern matching
Recursion over lists

Lists are the most important data type
in functional programming

```
[1, 2, 3, -42] :: [Integer]

[False] :: [Bool]

[’C’, ’h’, ’a’, ’r’] :: [Char]
```

=

```
"Char" :: String
```

because

```
type String = [Char]

[not, not] ::

[] :: [T]      -- empty list for any type T

[[True],[]] ::
```

Typing rule

If $e_1 :: T \quad \ldots \quad e_n :: T$
then $[e_1, \ldots, e_n] :: [T]$

Graphical notation:

$$\frac{e_1 :: T \quad \ldots \quad e_n :: T}{[e_1, \ldots, e_n] :: [T]}$$

`[True, 'c']` is not type-correct!!!

All elements in a list must have the same type

Test

```
(True, ’c’) ::

[(True, ’c’), (False, ’d’)] ::

([True, False], [’c’, ’d’]) ::
```

List ranges

```
[1 .. 3] = [1, 2, 3]
[3 .. 1] = []
['a' .. 'c'] = ['a', 'b', 'c']
```

Concatenation: ++

Concatenates two lists of the same type:

`[1, 2] ++ [3] = [1, 2, 3]`

~~`[1, 2] ++ ['a']`~~

4.1 List comprehension

Set comprehensions:

$$\{x^2 \mid x \in \{1, 2, 3, 4, 5\}\}$$

The set of all x^2 *such that* x *is an element of* $\{1, 2, 3, 4, 5\}$

List comprehension:

```
[ x ^ 2 | x <- [1 .. 5]]
```

The list of all `x^2` *such that* `x` *is an element of* `[1 .. 5]`

List comprehension — Generators

```
[ x ^ 2 | x <- [1 .. 5]]
= [1, 4, 9, 16, 25]

[ toLower c | c <- "Hello, World!"]
= "hello, world!"

[ (x, even x) | x <- [1 .. 3]]
= [(1, False), (2, True), (3, False)]

[ x+y | (x,y) <- [(1,2), (3,4), (5,6)]]
= [3, 7, 11]
```

pattern <- *list expression*
is called a *generator*

Precise definition of *pattern* later.

List comprehension — Tests

```
[ x*x | x <- [1 .. 5], odd x]
= [1, 9, 25]

[ x*x | x <- [1 .. 5], odd x, x > 3]
= [25]

[ toLower c | c <- "Hello, World!", isAlpha c]
= "helloworld"
```

Boolean expressions are called *tests*

Defining functions by list comprehension

Example

```
factors :: Int -> [Int]
factors n  =  [m | m <- [1 .. n], n ‘mod‘ m == 0]
```

⟹ `factors 15 = [1, 3, 5, 15]`

```
prime :: Int -> Bool
prime n  =  factors n == [1,n]
```

⟹ `prime 15 = False`

```
primes :: Int -> [Int]
primes n  =  [p | p <- [1 .. n], prime p]
```

⟹ `primes 100 = [2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,`

List comprehension — General form

$$[\ expr\ \mid\ E_1,\ \ldots,\ E_n\ ]$$

where $expr$ is an expression and each E_i is a generator or a test

Multiple generators

```
[(i,j) | i <- [1 .. 2], j <- [7 .. 9]]
```

```
= [(1,7), (1,8), (1,9), (2,7), (2,8), (2,9)]
```

Analogy: each generator is a `for` loop:

```
for all i <- [1 .. 2]
  for all j <- [7 .. 9]
    ...
```

Key difference:

Loops *do* something
Expressions *produce* something

Dependent generators

```
[(i,j) | i <- [1 .. 3], j <- [i .. 3]]
= [(1,j) | j <- [1..3]] ++
  [(2,j) | j <- [2..3]] ++
  [(3,j) | j <- [3..3]]
= [(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)]
```

The meaning of list comprehensions

$$\texttt{[}e \texttt{ | } x \texttt{ <- [}a_1\texttt{,...,}a_n\texttt{]]}$$
$$\texttt{= (let } x \texttt{ = } a_1 \texttt{ in [}e\texttt{]) ++ } \cdots \texttt{ ++ (let } x \texttt{ = } a_n \texttt{ in [}e\texttt{])}$$

$$\texttt{[}e \texttt{ | } b\texttt{]}$$
$$\texttt{= if } b \texttt{ then [}e\texttt{] else []}$$

$$\texttt{[}e \texttt{ | } x \texttt{ <- [}a_1\texttt{,...,}a_n\texttt{], } \overline{E}\texttt{]}$$
$$\texttt{= (let } x \texttt{ = } a_1 \texttt{ in [}e \texttt{ | } \overline{E}\texttt{]) ++ } \cdots \texttt{ ++}$$
$$\texttt{(let } x \texttt{ = } a_n \texttt{ in [}e \texttt{ | } \overline{E}\texttt{])}$$

$$\texttt{[}e \texttt{ | } b\texttt{, } \overline{E}\texttt{]}$$
$$\texttt{= if } b \texttt{ then [}e \texttt{ | } \overline{E}\texttt{] else []}$$

Example: `concat`

```
concat xss  =  [x | xs <- xss, x <- xs]

concat [[1,2], [4,5,6]]
= [x | xs <- [[1,2], [4,5,6]], x <- xs]
= [x | x <- [1,2]] ++ [x | x <- [4,5,6]]
= [1,2] ++ [4,5,6]
= [1,2,4,5,6]
```

What is the type of `concat`?

```
[[a]] -> [a]
```

4.2 Generic functions: Polymorphism

Polymorphism = one function can have many types

Example

```
length :: [Bool] -> Int
length :: [Char] -> Int
length :: [[Int]] -> Int
```

⋮

The most general type:

```
length :: [a] -> Int
```

where `a` is a *type variable*

⟹ `length :: [`*T*`] -> Int` *for all types T*

Type variable syntax

Type variables must start with a lower-case letter

Typically: `a`, `b`, `c`, . . .

Two kinds of polymorphism

Subtype polymorphism as in Java:

$$\frac{f :: T \to U \qquad T' \leq T}{f :: T' \to U}$$

(remember: horizontal line = implication)

Parametric polymorphism as in Haskell:

Types may contain type variables ("parameters")

$$\frac{f :: T}{f :: T[U/a]}$$

where $T[U/a]$ = "T with a replaced by U"
Example: $(a \to a)[Bool/a] \; = \; Bool \to Bool$

(Often called *ML-style polymorphism*)

Defining polymorphic functions

```
id :: a -> a
id x  =  x

fst :: (a,b) -> a
fst (x,y)  =  x

swap :: (a,b) -> (b,a)
swap (x,y)  =  (y,x)

silly :: Bool -> a -> Char
silly x y  =  if x then ’c’ else ’d’

silly2 :: Bool -> Bool -> Bool
silly2 x y  =  if x then x else y
```

Polymorphic list functions from the Prelude

```
length :: [a] -> Int
length [5, 1, 9] = 3

(++) :: [a] -> [a] -> [a]
[1, 2] ++ [3, 4] = [1, 2, 3, 4]

reverse :: [a] -> [a]
reverse [1, 2, 3] = [3, 2, 1]

replicate :: Int -> a -> [a]
replicate 3 'c' = "ccc"
```

Polymorphic list functions from the Prelude

```
head, last :: [a] -> a
head "list" = ’l’,    last "list" = ’t’

tail, init :: [a] -> [a]
tail "list" = "ist",    init "list" = "lis"

take, drop :: Int -> [a] -> [a]
take 3 "list" = "lis",    drop 3 "list" = "t"

-- A property:
prop_take_drop xs  =
  take n xs ++ drop n xs == xs
```

Polymorphic list functions from the Prelude

```
concat :: [[a]] -> [a]
concat [[1, 2], [3, 4], [0]] = [1, 2, 3, 4, 0]

zip :: [a] -> [b] -> [(a,b)]
zip [1,2] "ab" = [(1, ’a’), (2, ’b’)]

unzip :: [(a,b)] -> ([a],[b])
unzip [(1, ’a’), (2, ’b’)] = ([1,2], "ab")

-- A property
prop_zip xs ys = length xs == length ys ==>
  unzip(zip xs ys) == (xs, ys)
```

Haskell libraries

- Prelude and much more
- Hoogle — searching the Haskell libraries
- Hackage — a collection of Haskell packages

See Haskell pages and Thompson's book for more information.

Further list functions from the Prelude

```
and :: [Bool] -> Bool
and [True, False, True] = False

or :: [Bool] -> Bool
or [True, False, True] = True

-- For numeric types a:
sum, product :: [a] -> a
sum [1, 2, 2] = 5,    product [1, 2, 2] = 4
```

What exactly is the type of `sum`, `prod`, `+`, `*`, `==`, . . . ???

Polymorphism versus Overloading

Polymorphism: one definition, many types

Overloading: different definition for different types

Example

Function `(+)` is overloaded:

- on type `Int`: built into the hardware
- on type `Integer`: realized in software

So what is the type of `(+)` ?

Numeric types

```
(+) :: Num a => a -> a -> a
```

Function `(+)` has type `a -> a -> a` for any type of class `Num`

- Class `Num` is the class of *numeric types*.
- Predefined numeric types: `Int`, `Integer`, `Float`
- Types of class Num offer the basic arithmetic operations:

```
(+) :: Num a => a -> a -> a
(-) :: Num a => a -> a -> a
(*) :: Num a => a -> a -> a
⋮
sum, product :: Num a => [a] -> a
```

Other important type classes

- The class `Eq` of *equality types*, i.e. types that posess

  ```
  (==) ::  Eq a => a -> a -> Bool
  (/=) ::  Eq a => a -> a -> Bool
  ```

  Most types are of class `Eq`. Exception:

- The class `Ord` of *ordered types*, i.e. types that posess

  ```
  (<)  ::  Ord a => a -> a -> Bool
  (<=) ::  Ord a => a -> a -> Bool
  ```

More on type classes later. Don't confuse with OO classes.

Warning: `== []`

```
null xs  =  xs == []
```

Why?

`==` on `[a]` must (potentially) call `==` on `a`

Better:

```
null :: [a] -> Bool
null []  =  True
null _   =  False
```

In Prelude!

Warning: QuickCheck and polymorphism

QuickCheck does not work well on polymorphic properties

Example

QuickCheck does not find a counterexample to

```
prop_reverse :: [a] -> Bool
prop_reverse xs  =  reverse xs == xs
```

The solution: specialize the polymorphic property, e.g.

```
prop_reverse :: [Int] -> Bool
prop_reverse xs  =  reverse xs == xs
```

Now QuickCheck works

Conditional properties have result type `Property`

Example

```
prop_rev10 :: [Int] -> Property
prop_rev10 xs =
  length xs <= 10 ==> reverse(reverse xs) == xs
```

4.3 Case study: Pictures

```
type Picture = [String]
```

```
uarr ::  Picture
uarr =
 ["  #  ",
  " ### ",
  "#####",
  "  #  ",
  "  #  ",
```

```
larr ::  Picture
larr =
 ["  #  ",
  " ##  ",
  "#####",
  " ##  ",
  "  #  ",
```

```
flipH :: Picture -> Picture
flipH  =  reverse

flipV :: Picture -> Picture
flipV pic  =  [ reverse line | line <- pic]

rarr :: Picture
rarr  =  flipV larr

darr :: Picture
darr  =  flipH uarr

above :: Picture -> Picture -> Picture
above  =  (++)

beside :: Picture -> Picture -> Picture
beside pic1 pic2 = [ l1 ++ l2 | (l1,l2) <- zip pic1 pic2]
```

`PictureDemo.hs`

Chessboards

```
bSq  =  replicate 5 (replicate 5 ’#’)

wSq  =  replicate 5 (replicate 5 ’ ’)

alterH :: Picture -> Picture -> Int -> Picture
alterH pic1 pic2 1  =  pic1
alterH pic1 pic2 n  =  pic1 ‘beside‘ alterH pic2 pic1 (n-1

alterV :: Picture -> Picture -> Int -> Picture
alterV pic1 pic2 1  =  pic1
alterV pic1 pic2 n  =  pic1 ‘above‘ alterV pic2 pic1 (n-1)

chessboard :: Int -> Picture
chessboard n  =  alterV bw wb n where
  bw = alterH bSq wSq n
  wb = alterH wSq bSq n
```

Exercise

Ensure that the lower left square of `chesboard n` is always black.

4.4 Pattern matching

Every list can be constructed from `[]`
by repeatedly adding an element at the front
with the "cons" operator `(:) :: a -> [a] -> [a]`

syntactic sugar	in reality
`[3]`	`3 : []`
`[2, 3]`	`2 : 3 : []`
`[1, 2, 3]`	`1 : 2 : 3 : []`
$[x_1, \ldots, x_n]$	$x_1 : \ldots : x_n : []$

Note: $x : y : zs = x : (y : zs)$
`(:)` associates to the right

$\Longrightarrow$

Every list is either

`[]` or of the form

x : xs where

x is the *head* (first element, *Kopf*), and

xs is the *tail* (rest list, *Rumpf*)

`[]` and `(:)` are called *constructors*
because every list can be constructed uniquely from them.

$\Longrightarrow$

Every non-empty list can be decomposed uniquely into head and tail.

Therefore these definitions make sense:

```
head (x : xs)  =  x
tail (x : xs)  =  xs
```

`(++)` is not a constructor:

`[1,2,3]` is not uniquely constructable with `(++)`:
`[1,2,3] = [1] ++ [2,3] = [1,2] ++ [3]`

Therefore this definition does not make sense:

```
nonsense (xs ++ ys) = length xs - length ys
```

Patterns

Patterns are expressions
consisting only of constructors and variables.
No variable must occur twice in a pattern.

$\Longrightarrow$ Patterns allow unique decomposition = *pattern matching*.

A *pattern* can be

- a variable such as `x` or a wildcard `_` (underscore)
- a literal like `1`, `'a'`, `"xyz"`, ...
- a tuple `(`p_1`, ..., `p_n`)` where each p_i is a pattern
- a constructor pattern $C\ p_1\ \ldots\ p_n$
  where C is a constructor and each p_i is a pattern

Note: `True` and `False` are constructors, too!

Function definitions by pattern matching

Example

```
head :: [a] -> a
head (x : _)  =  x

tail :: [a] -> [a]
tail (_ : xs)  =  xs

null :: [a] -> Bool
null []        =  True
null (_ : _)   =  False
```

Function definitions by pattern matching

$$
\begin{array}{l}
f\ pat_1\ =\ e_1 \\
\vdots \\
f\ pat_n\ =\ e_n
\end{array}
$$

If f has multiple arguments:

$$
\begin{array}{l}
f\ pat_{11} \ldots\ pat_{1k}\ =\ e_1 \\
\vdots
\end{array}
$$

Conditional equations:

$$f\ \ patterns\ \ |\ \ condition\ \ =\ \ e$$

When f is called, the equations are tried in the given order

Function definitions by pattern matching

Example (contrived)

```
true12 :: [Bool] -> Bool
true12 (True : True : _)  =  True
true12 _  =  False

same12 :: Eq a => [a] -> [a] -> Bool
same12 (x : _) (_ : y : _)  =  x == y

asc3 :: Ord a => [a] -> Bool
asc3 (x : y : z : _)  =  x < y && y < z
asc3 (x : y : _)  =  x < y
asc3 _  =  True
```

4.5 Recursion over lists

Example

```
length []        =  0
length (_ : xs)  =  length xs + 1

reverse []       =  []
reverse (x : xs) =  reverse xs ++ [x]

sum :: Num a => [a] -> a
sum []       =  0
sum (x : xs) =  x + sum xs
```

Primitive recursion on lists:

```
f []        =  base     -- base case
f (x : xs)  =  rec      -- recursive case
```

- *base*: no call of f
- *rec*: only call(s) $f\ xs$

f may have additional parameters.

Finding primitive recursive definitions

Example

```
concat :: [[a]] -> [a]
concat []  =  []
concat (xs : xss)  =  xs ++ concat xss

(++) :: [a] -> [a] -> [a]
[] ++ ys  =  ys
(x:xs) ++ ys  =  x : (xs ++ ys)
```

Insertion sort

Example

```
inSort :: Ord a => [a] -> [a]
inSort []       =  []
inSort (x:xs)   =  ins x (inSort xs)

ins :: Ord a => a -> [a] -> [a]
ins x []  =  [x]
ins x (y:ys) | x <= y      =  x : y : ys
             | otherwise   =  y : ins x ys
```

Beyond primitive recursion: Multiple arguments

Example

```
zip :: [a] -> [b] -> [(a,b)]
zip (x:xs) (y:ys)  =  (x,y) : zip xs ys
zip _ _   =  []
```

Alternative definition:

```
zip’ [] []  =  []
zip’ (x:xs) (y:ys)  =  (x,y) : zip’ xs ys
```

`zip’` is undefined for lists of different length!

Beyond primitive recursion: Multiple arguments

Example

```
take :: Int -> [a] -> [a]
take 0 _   =  []
take _ []  =  []
take i (x:xs) | i>0  =  x : take (i-1) xs
```

General recursion: Quicksort

Example

```
quicksort :: Ord a => [a] -> [a]
quicksort []  =  []
quicksort (x:xs)  =
  quicksort below ++ [x] ++ quicksort above
    where
      below = [y | y <- xs, y <= x]
      above = [y | y <- xs, x < y]
```

Accumulating parameter

Idea: Result is accumulated in parameter and returned later

Example: list of all (maximal) ascending sublists in a list

`ups [3,0,2,3,2,4] = [[3], [0,2,3], [2,4]]`

```
ups2 :: Ord a => [a] -> [a] -> [[a]]
-- 1st param: input list
-- 2nd param: partial ascending sublist (reversed)
ups2 (x:xs) (y:ys)
  | x >= y       =  ups2 xs (x:y:ys)
  | otherwise    =  reverse (y:ys) : ups2 (x:xs) []
ups2 (x:xs) []   = ups2 xs [x]
ups2 []     ys   = [reverse ys]

ups :: Ord a => [a] -> [[a]]
ups xs = ups2 xs []
```

How can we `quickCheck` the result of `ups`?

Convention

Identifiers of list type end in ‘s’:

`xs`, `ys`, `zs`, . . .

Mutual recursion

Example

```
even :: Int -> Bool
even n  =  n == 0 || n > 0 && odd (n-1) || odd (n+1)

odd :: Int -> Bool
odd n  =  n /= 0 && (n > 0 && even (n-1) || even (n+1))
```

Scoping by example

```
x = y + 5
y = x + 1 where x = 7
f y = y + x
```

```
> f 3
```

Binding occurrence

Bound occurrence

Scope of binding

Scoping by example

```
x = y + 5
y = x + 1 where x = 7
f y = y + x
```

```
> f 3
```

Binding occurrence
Bound occurrence
Scope of binding

Scoping by example

```
x = y + 5
y = x + 1 where x = 7
f y = y + x
```

`> f 3`

Binding occurrence

Bound occurrence

Scope of binding

Scoping by example

```
x = y + 5
y = x + 1 where x = 7
f y = y + x
```

```
> f 3
```

Binding occurrence

Bound occurrence

Scope of binding

Scoping by example

```
x = y + 5
y = x + 1 where x = 7
f y = y + x
```

```
> f 3
```

Binding occurrence

Bound occurrence

Scope of binding

Scoping by example

Summary:

- Order of definitions is irrelevant
- Parameters and `where`-defs are local to each equation

5. Proofs

Proving properties
Definedness

Aim

Guarentee functional (I/O) properties of software

- Testing can guarantee properties for some inputs.
- Mathematical proof can guarantee properties for all inputs.

QuickCheck is good, proof is better

Beware of bugs in the above code;
I have only proved it correct, not tried it.

Donald E. Knuth, 1977

5.1 Proving properties

What do we prove?

Equations `e1 = e2`

How do we prove them?

By using defining equations `f p = t`

A first, simple example

Remember:

```
      [] ++ ys = ys
(x:xs) ++ ys = x : (xs ++ ys)
```

Proof of `[1,2] ++ [] = [1] ++ [2]`:

```
1:2:[] ++ []
= 1 : (2:[] ++ [])     -- by def of ++
= 1 : 2 : ([] ++ [])   -- by def of ++
= 1 : 2 : []           -- by def of ++
= 1 : ([] ++ 2:[])     -- by def of ++
= 1:[] ++ 2:[]         -- by def of ++
```

Observation: first used equations from left to right (ok),
then from right to left (strange!)

A more natural proof of `[1,2] ++ [] = [1] ++ [2]`:

```
1:2:[] ++ []
= 1 : (2:[] ++ [])     -- by def of ++
= 1 : 2 : ([] ++ [])   -- by def of ++
= 1 : 2 : []           -- by def of ++

1:[] ++ 2:[]
= 1 : ([] ++ 2:[])     -- by def of ++
= 1 : 2 : []           -- by def of ++
```

Proofs of `e1 = e2` are often better presented as two reductions to some expression `e`:

```
e1 = ...   = e
e2 = ...   = e
```

Fact If an equation does not contain any variables, it can be proved by evaluating both sides separately and checking that the result is identical.

But how to prove equations with variables, for example *associativity* of ++:

```
(xs ++ ys) ++ zs = xs ++ (ys ++ zs)
```

Properties of recursive functions are proved by induction

Induction on natural numbers: see Diskrete Strukturen

Induction on lists: here and now

Structural induction on lists

To prove property `P(xs)` for all finite lists `xs`

Base case: Prove `P([])` and

Induction step: Prove `P(xs)` implies `P(x:xs)`

$$\underset{\textit{induction hypothesis (IH)}}{\overset{\uparrow}{\texttt{P(xs)}}} \quad \text{implies} \quad \underset{\text{new variable } \texttt{x}}{\overset{\uparrow}{\texttt{P(x:xs)}}}$$

This is called *structural induction* on `xs`.
It is a special case of induction on the length of `xs`.

Example: associativity of ++

Lemma app_assoc: `(xs ++ ys) ++ zs = xs ++ (ys ++ zs)`
Proof by structural induction on `xs`

Base case:
To show: `([] ++ ys) ++ zs = [] ++ (ys ++ zs)`

```
([] ++ ys) ++ zs
= ys ++ zs              -- by def of ++
= [] ++ (ys ++ zs)      -- by def of ++
```

Induction step:
To show: `((x:xs) ++ ys) ++ zs = (x:xs) ++ (ys ++ zs)`

```
((x:xs) ++ ys) ++ zs
= (x : (xs ++ ys)) ++ zs  -- by def of ++
= x : ((xs ++ ys) ++ zs)  -- by def of ++
= x : (xs ++ (ys ++ zs))  -- by IH
(x:xs) ++ (ys ++ zs)
= x : (xs ++ (ys ++ zs))  -- by def of ++
```

Induction template

Lemma `P(xs)`
Proof by structural induction on `xs`

Base case:
To show: `P([])`

Proof of `P([])`

Induction step:
To show: `P(x:xs)`

Proof of `P(x:xs)` using IH `P(xs)`

Example: `length` of `++`

Lemma `length(xs ++ ys) = length xs + length ys`

Proof by structural induction on `xs`

Base case:

To show: `length ([] ++ ys) = length [] + length ys`

```
length ([] ++ ys)
= length ys              -- by def of ++
length [] + length ys
= 0 + length ys          -- by def of length
= length ys
```

Induction step:

To show: `length((x:xs)++ys) = length(x:xs) + length ys`

```
length((x:xs) ++ ys)
= length(x : (xs ++ ys))     -- by def of ++
= 1 + length(xs ++ ys)       -- by def of length
= 1 + length xs + length ys  -- by IH
length(x:xs) + length ys
= 1 + length xs + length ys  -- by def of length
```

Example: `reverse` of `++`

Lemma `reverse(xs ++ ys) = reverse ys ++ reverse xs`
Proof by structural induction on `xs`
Base case:
To show: `reverse ([] ++ ys) = reverse ys ++ reverse []`

```
reverse ([] ++ ys)
= reverse ys                  -- by def of ++
reverse ys ++ reverse []
= reverse ys ++ []            -- by def of reverse
= reverse ys                  -- by Lemma app_Nil2
```

Lemma app_Nil2: `xs ++ [] = xs`
Proof exercise

Induction step:

To show: `reverse((x:xs)++ys) = reverse ys ++ reverse(x:xs)`

```
reverse((x:xs) ++ ys)
= reverse(x : (xs ++ ys))           -- by def of ++
= reverse(xs ++ ys) ++ [x]          -- by def of reverse
= (reverse ys ++ reverse xs) ++ [x]  -- by IH
= reverse ys ++ (reverse xs ++ [x])  -- by Lemma app_assoc
```

```
reverse ys ++ reverse(x:xs)
= reverse ys ++ (reverse xs ++ [x])  -- by def of reverse
```

Proof heuristic

- Try QuickCheck
- Try to evaluate both sides to common term
- Try induction
    - Base case: reduce both sides to a common term using function defs and lemmas
    - Induction step: reduce both sides to a common term using function defs, IH and lemmas
- If base case or induction step fails:
  conjecture, prove and use new lemmas

Two further tricks

- Proof by cases
- Generalization

Example: proof by cases

```
rem x [] = []
rem x (y:ys) | x==y      = rem x ys
             | otherwise = y : rem x ys
```

Lemma `rem z (xs ++ ys) = rem z xs ++ rem z ys`

Proof by structural induction on `xs`

Base case:

To show: `rem z ([] ++ ys) = rem z [] ++ rem z ys`

```
 rem z ([] ++ ys)
 = rem z ys             -- by def of ++
 rem z [] ++ rem z ys
 = rem z ys             -- by def of rem and ++
```

```
rem x [] = []
rem x (y:ys) | x==y      = rem x ys
             | otherwise = y : rem x ys
```

Induction step:

To show: `rem z ((x:xs)++ys) = rem z (x:xs) ++ rem z ys`

Proof by cases:

Case `z == x`:

```
rem z ((x:xs) ++ ys)
= rem z (xs ++ ys)          -- by def of ++ and rem
= rem z xs ++ rem z ys      -- by IH
rem z (x:xs) ++ rem z ys
= rem z xs ++ rem z ys      -- by def of rem
```

Case `z /= x`:

```
rem z ((x:xs) ++ ys)
= x : rem z (xs ++ ys)          -- by def of ++ and rem
= x : (rem z xs ++ rem z ys)    -- by IH
rem z (x:xs) ++ rem z ys
= x : (rem z xs ++ rem z ys)    -- by def of rem and ++
```

Inefficiency of `reverse`

```
reverse [1,2,3]
= reverse [2,3] ++ [1]
= (reverse [3] ++ [2]) ++ [1]
= ((reverse [] ++ [3]) ++ [2]) ++ [1]
= (([] ++ [3]) ++ [2]) ++ [1]
= ([3] ++ [2]) ++ [1]
= (3 : ([] ++ [2])) ++ [1]
= [3,2] ++ [1]
= 3 : ([2] ++ [1])
= 3 : (2 : ([] ++ [1]))
= [3,2,1]
```

An improvement: `itrev`

```
itrev :: [a] -> [a] -> [a]
itrev [] xs     = xs
itrev (x:xs) ys = itrev xs (x:ys)
```

```
itrev [1,2,3] []
= itrev [2,3] [1]
= itrev [3] [2,1]
= itrev [] [3,2,1]
= [3,2,1]
```

Proof attempt

Lemma `itrev xs [] = reverse xs`

Proof by structural induction on `xs`

Induction step fails:

To show: `itrev (x:xs) [] = reverse xs`

```
itrev (x:xs) []
= itrev xs [x]        -- by def of itrev
reverse (x:xs)
= reverse xs ++ [x]  -- by def of reverse
```

Problem: IH not applicable because too specialized: `[]`

Generalization

Lemma `itrev xs ys = reverse xs ++ ys`

Proof by structural induction on `xs`

Induction step:

To show: `itrev (x:xs) ys = reverse (x:xs) ++ ys`

```
itrev (x:xs) ys
= itrev xs (x:ys)                -- by def of itrev
= reverse xs ++ (x:ys)           -- by IH
reverse (x:xs) ++ ys
= (reverse xs ++ [x]) ++ ys      -- by def of reverse
= reverse xs ++ ([x] ++ ys)      -- by Lemma app_assoc
= reverse xs ++ (x:ys)           -- by def of ++
```

Note: IH is used with `x:ys` instead of `ys`

When using the IH, variables may be replaced by arbitrary expressions, only the induction variable must stay fixed.

Justification: all variables are implicitly $\forall$-quantified, except for the induction variable.

Induction on the length of a list

```
qsort :: Ord a => [a] -> [a]
qsort []       =  []
qsort (x:xs)  = qsort below ++ [x] ++ qsort above
    where below = [y | y <- xs, y <= x]
          above = [z | y <- xs, x < z]
```

Lemma `qsort xs` is sorted

Proof by induction on the length of the argument of `qsort`.

Induction step: In the call `qsort (x:xs)` we have `length below <= length xs < length(x:xs)` (also for above).
Therefore `qsort below` and `qsort above` are sorted by IH.
By construction `below` contains only elements `(<=x)`.
Therefore `qsort below` contains only elements `(<=x)` (proof!).
Analogously for `above` and `(x<)`.
Therefore `qsort (x:xs)` is sorted.

Is that all? Or should we prove something else about sorting?

How about this sorting function?

```
superquicksort _ = []
```

Every element should occur as often in the output as in the input!

5.2 Definedness

Simplifying assumption, implicit so far:

No undefined values

Two kinds of undefinedness:

`head []`	raises exception
`f x = f x + 1`	does not terminate

Undefinedness can be handled, too.
But it complicates life

What is the problem?

Many familiar laws no longer hold unconditionally:

```
x - x = 0
```

is true only if `x` is a defined value.

Two examples:

- Not true: `head [] - head [] = 0`
- From the nonterminating definition
  `f x = f x + 1`
  we could conclude that $0 = 1$.

Termination

Termination of a function means termination for all inputs.

Restriction:

> The proof methods in this chapter assume that all recursive definitions under consideration terminate.

Most Haskell functions we have seen so far terminate.

How to prove termination

Example

```
reverse [] = []
reverse (x:xs) = reverse xs ++ [x]
```

terminates because ++ terminates and with each recursive call of `reverse`, the length of the argument becomes smaller.

A function `f :: T1 -> T` terminates
if there is a *measure function* `m :: T1 ->` $\mathbb{N}$ such that

- for every defining equation `f p = t`
- and for every recursive call `f r` in `t`: `m p > m r`.

Note:

- All primitive recursive functions terminate.
- `m` can be defined in Haskell or mathematics.
- The conditions above can be refined to take special Haskell features into account, eg sequential pattern matching.

More generally: `f :: T1 -> ... -> Tn -> T` terminates if there is a measure function `m :: T1 -> ... -> Tn -> ℕ` such that

- for every defining equation `f p1 ... pn = t`
- and for every recursive call `f r1 ... rn` in t: `m p1 ... pn > m r1 ... rn`.

Infinite values

Haskell allows infinite values, in particular infinite lists.

Example: `[1, 1, 1, ...]`

Infinite objects must be constructed by recursion:

```
ones = 1 : ones
```

Because we restrict to terminating definitions in this chapter, infinite values cannot arise.

Note:

- By termination of functions we really mean termination on *finite* values.
- For example `reverse` terminates only on finite lists.

This is fine because we can only construct finite values anyway.

How can infinite values be useful?
Because of “lazy evaluation”.
More later.

Exceptions

If we use arithmetic equations like `x - x = 0` unconditionally, we can "lose" exceptions:

`head xs - head xs = 0`

is only true if `xs /= []`

In such cases, we can prove equations `e1 = e2` that are only *partially correct*:

> If for some values for the variables in `e1` and `e2`
> `e1` and `e2` do not produce a runtime exception
> then they evaluate to the same value.

Summary

- In this chapter everything must terminate
- This avoids undefined and infinite values
- This simplifies proofs

6. Higher-Order Functions

- Applying functions to all elements of a list: `map`
- Filtering a list: `filter`
- Combining the elements of a list: `foldr`
- Lambda expressions
- Extensionality
- Curried functions
- More library functions
- Case study: Counting words

Recall [`Pic` is short for `Picture`]

```
alterH :: Pic -> Pic -> Int -> Pic
alterH pic1 pic2 1 = pic1
alterH pic1 pic2 n = beside pic1 (alterH pic2 pic1 (n-1))

alterV :: Pic -> Pic -> Int -> Pic
alterV pic1 pic2 1 = pic1
alterV pic1 pic2 n = above pic1 (alterV pic2 pic1 (n-1))
```

Very similar. Can we avoid duplication?

```
alt :: (Pic -> Pic -> Pic) -> Pic -> Pic -> Int -> Pic
alt f pic1 pic2 1 = pic1
alt f pic1 pic2 n = f pic1 (alt f pic2 pic1 (n-1))

alterH pic1 pic2 n = alt beside pic1 pic2 n

alterV pic1 pic2 n = alt above pic1 pic2 n
```

Higher-order functions:
Functions that take functions as arguments

`... -> (... -> ...) -> ...`

Higher-order functions capture patterns of computation

6.1 Applying functions to all elements of a list: `map`

Example

```
map even [1, 2, 3]
= [False, True, False]

map toLower "R2-D2"
= "r2-d2"

map reverse ["abc", "123"]
= ["cba", "321"]
```

What is the type of `map`?

```
map :: (a -> b) -> [a] -> [b]
```

`map`: The mother of all higher-order functions

Predefined in `Prelude`.
Two possible definitions:

```
map f xs  =  [ f x | x <- xs ]

map f []      =  []
map f (x:xs)  =  f x : map f xs
```

Evaluating `map`

```
map f []       =   []
map f (x:xs)   =   f x : map f xs

map sqr [1, -2]
= map sqr (1 : -2 : [])
= sqr 1 : map sqr (-2 : [])
= sqr 1 : sqr (-2) : (map sqr [])
= sqr 1 : sqr (-2) : []
= 1 : 4 : []
= [1, 4]
```

Some properties of `map`

```
length (map f xs) = length xs

map f (xs ++ ys) = map f xs ++ map f ys

map f (reverse xs) = reverse (map f xs)
```

Proofs by induction

QuickCheck and function variables

QuickCheck does not work automatically
for properties of function variables

It needs to know how to generate and print functions.

Cheap alternative: replace function variable by specific function(s)

Example

```
prop_map_even :: [Int] -> [Int] -> Bool
prop_map_even xs ys =
  map even (xs ++ ys) = map even xs ++ map even ys
```

6.2 Filtering a list: `filter`

Example

```
filter even [1, 2, 3]
= [2]

filter isAlpha "R2-D2"
= "RD"

filter null [[], [1,2], []]
= [[], []]
```

What is the type of `filter`?

```
filter :: (a -> Bool) -> [a] -> [a]
```

`filter`

Predefined in `Prelude`.
Two possible definitions:

```
filter p xs  =  [ x | x <- xs, p x ]

filter p []                    =  []
filter p (x:xs) | p x          =  x : filter p xs
                | otherwise    =  filter p xs
```

Some properties of `filter`

```
filter p (xs ++ ys) = filter p xs ++ filter p ys
```

```
filter p (reverse xs) = reverse (filter p xs)
```

Proofs by induction

6.3 Combining the elements of a list: `foldr`

Example

```
sum []      =  0
sum (x:xs)  =  x + sum xs
```

$$\texttt{sum}\ [x_1,\ \ldots,\ x_n] = x_1 + \ldots + x_n + 0$$

```
concat []        =  []
concat (xs:xss)  =  xs ++ concat xss
```

$$\texttt{concat}\ [xs_1,\ \ldots,\ xs_n] = xs_1 \mathbin{+\!\!+} \ldots \mathbin{+\!\!+} xs_n \mathbin{+\!\!+} []$$

foldr

$$\texttt{foldr}\ (\oplus)\ z\ [x_1,\ \ldots,\ x_n] = x_1 \oplus \ldots \oplus x_n \oplus z$$

Defined in `Prelude`:

```
foldr :: (a -> a -> a) -> a -> [a] -> a
foldr f a []        =  a
foldr f a (x:xs)    =  x `f` foldr f a xs
```

Applications:

```
sum xs  =  foldr (+) 0 xs

concat xss =  foldr (++) [] xss
```

What is the most general type of `foldr`?

`foldr`

```
foldr f a []      =  a
foldr f a (x:xs)  =  x `f` foldr f a xs
```

`foldr f a` replaces
`(:)` by `f` and
`[]` by `a`

Evaluating `foldr`

```
foldr f a []        =  a
foldr f a (x:xs)    =  x `f` foldr f a xs
```

```
foldr (+) 0 [1, -2]
= foldr (+) 0 (1 : -2 : [])
= 1 + foldr (+) 0 (-2 : [])
= 1 + -2 + (foldr (+) 0 [])
= 1 + (-2 + 0)
= -1
```

More applications of `foldr`

```
product xs   =  foldr  (*)   1      xs
and xs       =  foldr  (&&)  True   xs
or xs        =  foldr  (||)  False  xs
inSort xs    =  foldr  ins   []     xs
```

Quiz

What is

```
foldr (:) ys xs
```

Example: `foldr (:) ys (1:2:3:[]) = 1:2:3:ys`

```
foldr (:) ys xs = ???
```

Proof by induction on `xs` (Exercise!)

Defining functions via `foldr`

- means you have understood the art of higher-order functions
- allows you to apply properties of `foldr`

Example

If `f` is associative and `` a `f` x = x `` then
`` foldr f a (xs++ys) = foldr f a xs `f` foldr f a ys ``.

Proof by induction on `xs`. Induction step:

```
foldr f a ((x:xs) ++ ys) = foldr f a (x : (xs++ys))
= x `f` foldr f a (xs++ys)
= x `f` (foldr f a xs `f` foldr f a ys)      -- by IH
foldr f a (x:xs) `f` foldr f a ys
= (x `f` foldr f a xs) `f` foldr f a ys
= x `f` (foldr f a xs `f` foldr f a ys)      -- by assoc.
```

Therefore, if `g xs = foldr f a xs`,
then `` g (xs ++ ys) = g xs `f` g ys ``.

Therefore `sum (xs++ys) = sum xs + sum ys`,
`product (xs++ys) = product xs * product ys`, ...

6.4 Lambda expressions

Consider

```
squares xs  =  map sqr xs  where  sqr x = x * x
```

Do we really need to define `sqr` explicitly? No!

```
\x -> x * x
```

is the anonymous function with

formal parameter `x` and result `x * x`

In mathematics: $x \mapsto x * x$

Evaluation:

```
(\x -> x * x) 3 = 3 * 3 = 9
```

Usage:

```
squares xs  =  map (\x -> x * x) xs
```

Terminology

`(\x -> e1) e2`

x: formal parameter
e_1: result
e_2: actual parameter

Why "lambda"?

The logician Alonzo Church invented *lambda calculus* in the 1930s

Logicians write $\lambda x.\ e$ instead of `\x -> e`

Typing lambda expressions

Example

`(\x -> x > 0) :: Int -> Bool`

because `x :: Int` implies `x > 0 :: Bool`

The general rule:

$$\texttt{(\textbackslash}x\ \texttt{->}\ e\texttt{)} :: T_1\ \texttt{->}\ T_2$$

$$\text{if } x :: T_1 \text{ implies } e :: T_2$$

Sections of infix operators

```
(+ 1)   means   (\x -> x + 1)
(2 *)   means   (\x -> 2 * x)
(2 ^)   means   (\x -> 2 ^ x)
(^ 2)   means   (\x -> x ^ 2)
```

etc

Example

```
squares xs  =  map (^ 2) xs
```

List comprehension

Just syntactic sugar for combinations of `map`

```
[ f x | x <- xs ]     =     map f xs
```

`filter`

```
[ x | x <- xs, p x ]     =     filter p xs
```

and `concat`

```
            [f x y | x <- xs, y <- ys] =
concat (                                  )
```

6.5 Extensionality

Two functions are equal
if for all arguments they yield the same result

f, g `::` T_1 `->` T:

$$\frac{\forall a.\ f\ a = g\ a}{f = g}$$

f, g `::` T_1 `->` T_2 `->` T:

$$\frac{\forall a, b.\ f\ a\ b = g\ a\ b}{f = g}$$

6.6 Curried functions

A trick (re)invented by the logician Haskell Curry

Example

```
f :: Int -> Int -> Int        f :: Int -> (Int -> Int)
f x y  =  x+y                 f x  =  \y -> x+y
```

Both mean the same:

```
f a b                         (f a) b
= a + b                       = (\y -> a + y) b
                              = a + b
```

The trick: any function of two arguments
can be viewed as a function of the first argument
that returns a function of the second argument

In general

Every function is a function of one argument
(which may return a function as a result)

$$T_1 \texttt{ -> } T_2 \texttt{ -> } T$$

is just syntactic sugar for

$$T_1 \texttt{ -> } (T_2 \texttt{ -> } T)$$

$$f\ e_1\ e_2$$

is just syntactic sugar for

$$\underbrace{(f\ e_1)}_{::T_2 \texttt{ -> } T}\ e_2$$

Analogously for more arguments

`->` is not associative:

$$T_1 \texttt{ -> } (T_2 \texttt{ -> } T) \neq (T_1 \texttt{ -> } T_2) \texttt{ -> } T$$

Example

```
f :: Int -> (Int -> Int)      g :: (Int -> Int) -> Int
f x y  =  x + y               g h  =  h 0 + 1
```

Application is not associative:

$$(f\ e_1)\ e_2 \neq f\ (e_1\ e_2)$$

Example

`(f 3) 4` $\neq$ `f (3 4)`   `g (id abs)` $\neq$ `(g id) abs`

Quiz

```
head tail xs
```

Correct?

Partial application

Every function of n parameters
can be applied to less than n arguments

Example

Instead of `sum xs  =  foldr (+) 0 xs`

just define `sum  =  foldr (+) 0`

In general:

If f `::` T_1 `-> ... ->` T_n `->` T

and a_1 `::` T_1`, ...,` a_m `::` T_m and $m \leq n$

then $f\ a_1 \ldots a_m$ `::` T_{m+1} `-> ... ->` T_n `->` T

6.7 More library functions

```
(.) :: (b -> c) -> (a -> b) ->
f . g  =  \x -> f (g x)
```

Example

```
head2  =  head . tail

head2 [1,2,3]
= (head . tail) [1,2,3]
= (\x -> head (tail x)) [1,2,3]
= head (tail [1,2,3])
= head [2,3]
= 2
```

```
const :: a -> (b -> a)
const x  =  \ _ -> x

curry :: ((a,b) -> c) -> (a -> b -> c)
curry f  =  \ x y -> f(x,y)

uncurry :: (a -> b -> c) -> ((a,b) -> c)
uncurry f  =  \(x,y) -> f x y
```

```
all :: (a -> Bool) -> [a] -> Bool
all p xs  =  and [p x | x <- xs]
```

Example

```
all (>1) [0, 1, 2]
= False
```

```
any :: (a -> Bool) -> [a] -> Bool
any p  =  or [p x | x <- xs]
```

Example

```
any (>1) [0, 1, 2]
= True
```

```
takeWhile :: (a -> Bool) -> [a] -> [a]
takeWhile p []      = []
takeWhile p (x:xs)
    | p x           = x : takeWhile p xs
    | otherwise     = []
```

Example

```
takeWhile (not . isSpace) "the end"
= "the"
```

```
dropWhile :: (a -> Bool) -> [a] -> [a]
dropWhile p []      = []
dropWhile p (x:xs)
    | p x           = dropWhile p xs
    | otherwise     = x:xs
```

Example

```
dropWhile (not . isSpace) "the end"
= " end"
```

6.8 Case study: Counting words

Input: A string, e.g. `"never say never again"`

Output: A string listing the words in alphabetical order, together with their frequency,
e.g. `"again: 1\nnever: 2\nsay: 1\n"`

Function `putStr` yields

```
again: 1
never: 2
say: 1
```

Design principle:

> *Solve problem in a sequence of small steps*
> *transforming the input gradually into the output*

Unix pipes!

Step 1: Break input into words

$$\begin{tikzcd}
\texttt{"never say never again"} \arrow[d, "\text{function}"', "\texttt{words}"] \\
\texttt{["never", "say", "never", "again"]}
\end{tikzcd}$$

Predefined in Prelude

Step 2: Sort words

`["never", "say", "never", "again"]`

function ↓ `sort`

`["again", "never", "never", "say"]`

Predefined in `Data.List`

Step 3: Group equal words together

`["again", "never", "never", "say"]`

function ↓ `group`

`[["again"], ["never", "never"], ["say"]]`

Predefined in `Data.List`

Step 4: Count each group

```
[["again"], ["never", "never"], ["say"]]

      |
      |  map (\ws -> (head ws, length ws))
      ↓

[("again", 1), ("never", 2), ("say", 1)]
```

Step 5: Format each group

```
[("again", 1), ("never", 2), ("say", 1)]
```

↓ `map (\(w,n) -> (w ++ ": " ++ show n))`

```
["again: 1", "never: 2", "say: 1"]
```

Step 6: Combine the lines

`["again: 1", "never: 2", "say: 1"]`

function ↓ `unlines`

`"again: 1\nnever: 2\nsay: 1\n"`

Predefined in Prelude

The solution

```
countWords :: String -> String
countWords  =
  unlines
  . map (\(w,n) -> w ++ ": " ++ show n)
  . map (\ws -> (head ws, length ws))
  . group
  . sort
  . words
```

Merging `maps`

Can we merge two consecutive `map`s?

```
map f . map g = ???
```

The optimized solution

```
countWords :: String -> String
countWords  =
  unlines
  . map (\ws -> head ws ++ ": " ++ show(length ws))
  . group
  . sort
  . words
```

Proving `map f . map g = map (f.g)`

First we prove (why?)

```
map f (map g xs) = map (f.g) xs
```

by induction on `xs`:

- Base case:
  ```
  map f (map g []) = []
  map (f.g) [] = []
  ```
- Induction step:
  ```
  map f (map g (x:xs))
  = f (g x) : map f (map g xs)
  = f (g x) : map (f.g) xs      -- by IH
  map (f.g) (x:xs)
  = f (g x) : map (f.g) xs
  ```

$\Longrightarrow$ `(map f . map g) xs = map f (map g xs) = map (f.g) xs`

$\Longrightarrow$ `(map f . map g) = map (f.g)` by extensionality

7. Type Classes

Remember: type classes enable overloading

Example

```
elem ::
elem x  =  any (== x)
```

where `Eq` is the class of all types with ==

In general:

> *Type classes are collections of types*
> *that implement some fixed set of functions*

Haskell type classes are analogous to Java interfaces:
a set of function names with their types

Example

```
class Eq a where
  (==) :: a -> a -> Bool
```

Note: the type of `(==)` outside the class context is
`Eq a => a -> a -> Bool`

The general form of a class declaration:

```
class C a where
  f1 :: T1
  ...
  fn :: Tn
```

where the `Ti` may involve the type variable `a`

Instance

> *A type T is an instance of a class C*
> *if T supports all the functions of C.*
> *Then we write $C\ T$.*

Example

Type `Int` is an instance of class `Eq`, i.e., `Eq Int`

Therefore `elem :: Int -> [Int] -> Bool`

Warning Terminology clash:
Type T_1 is an *instance* of type T_2
if T_1 is the result of replacing type variables in T_2.
For example `(Bool,Int)` is an instance of `(a,b)`.

`instance`

The `instance` statement makes a type an instance of a class.

Example

```
instance Eq Bool where
  True  == True   =  True
  False == False  =  True
  _     == _      =  False
```

Instances can be constrained:

Example

```
instance Eq a => Eq [a] where
  []     == []       =  True
  (x:xs) == (y:ys)   =  x == y && xs == ys
  _      == _        =  False
```

Possibly with multiple constraints:

Example

```
instance (Eq a, Eq b) => Eq (a,b) where
  (x1,y1) == (x2,y2)  =  x1 == x2 && y1 == y2
```

The general form of the `instance` statement:

```
instance (context) => C T where
  definitions
```

T	is a type
context	is a list of assumptions $C_i \ T_i$
definitions	are definitions of the functions of class C

Subclasses

Example

```
class Eq a => Ord a where
  (<=), (<) :: a -> a -> Bool
```

Class `Ord` inherits all the operations of class `Eq`

Because `Bool` is already an instance of `Eq`,
we can now make it an instance of `Ord`:

```
instance Ord Bool where
  b1 <= b2  =  not b1 || b2
  b1 < b2  =  b1 <= b2 && not(b1 == b2)
```

From the Prelude: `Eq`, `Ord`, `Show`

```
class Eq a where
  (==), (/=) :: a -> a -> Bool
  -- default definition:
  x /= y  =  not(x==y)

class Eq a => Ord a where
  (<=), (<), (>=), (>) :: a -> a -> Bool
  -- default definitions:
  x < y  =  x <= y && x /= y
  x > y  =  y < x
  x >= y =  y <= x

class Show a where
  show :: a -> String
```

8. Algebraic `data` Types

`data` by example
The general case
Case study: boolean formulas
Structural induction

So far: no really new types,
just compositions of existing types

Example: `type String = [Char]`

Now: `data` defines *new* types

Introduction by example: From enumerated types
to recursive and polymorphic types

8.1 `data` by example

Bool

From the Prelude:

```
data Bool = False | True

not :: Bool -> Bool
not False  =  True
not True   =  False

(&&) :: Bool -> Bool -> Bool
False && q  =  False
True  && q  =  q

(||) :: Bool -> Bool -> Bool
False || q  = q
True  || q  = True
```

`deriving`

```
instance Eq Bool where
  True  == True   =  True
  False == False  =  True
  _     == _      =  False

instance Show Bool where
  show True   =  "True"
  show False  =  "False"
```

Better: let Haskell write the code for you:

```
data Bool = False | True
            deriving (Eq, Show)
```

`deriving` supports many more classes: `Ord`, `Read`, . . .

Warning

Do not forget to make your `data` types instances of `Show`

Otherwise Haskell cannot even print values of your type

Warning

QuickCheck does not automatically work for `data` types

You have to write your own test data generator. Later.

Season

```
data Season = Spring | Summer | Autumn | Winter
              deriving (Eq, Show)

next :: Season -> Season
next Spring = Summer
next Summer = Autumn
next Autumn = Winter
next Winter = Spring
```

Shape

```
type Radius = Float
type Width  = Float
type Height = Float
```

```
data Shape = Circle Radius | Rect Width Height
             deriving (Eq, Show)
```

Some values of type Shape: `Circle 1.0`

`Rect 0.9 1.1`

`Circle (-2.0)`

```
area :: Shape -> Float
area (Circle r)  =  pi * r^2
area (Rect w h)  =  w * h
```

Maybe

From the Prelude:

```
data Maybe a = Nothing | Just a
               deriving (Eq, Show)
```

Some values of type Maybe:

```
Nothing ::    Maybe a
Just True ::  Maybe Bool
Just "?" ::   Maybe String
```

```
lookup :: Eq a => a -> [(a,b)] -> Maybe b
lookup key []  =
lookup key ((x,y):xys)
  | key == x   =
  | otherwise  =
```

Nat

Natural numbers:

```
data Nat = Zero | Suc Nat
           deriving (Eq, Show)
```

Some values of type `Nat`:

```
Zero
Suc Zero
Suc (Suc Zero)
⋮
```

```
add :: Nat -> Nat -> Nat
add Zero n  =  n
add (Suc m) n  =

mul :: Nat -> Nat -> Nat
mul Zero n  =  Zero
mul (Suc m) n  =
```

Lists

From the Prelude:

```
data [a] = [] | (:) a [a]
           deriving Eq
```

The result of deriving `Eq`:

```
instance Eq a => Eq [a] where
  []     == []       =  True
  (x:xs) == (y:ys)   =  x == y && xs == ys
  _      == _        =  False
```

Defined explicitly:

```
instance Show a => Show [a] where
  show xs  =  "[" ++ concat cs ++ "]"
    where cs = Data.List.intersperse ", "  (map show xs)
```

Tree

```
data Tree a = Empty | Node a (Tree a) (Tree a)
              deriving (Eq, Show)
```

Some trees:

```
Empty
Node 1 Empty Empty
Node 1 (Node 2 Empty Empty) Empty
Node 1 Empty (Node 2 Empty Empty)
Node 1 (Node 2 Empty Empty) (Node 3 Empty Empty)
⋮
```

```
-- assumption: < is a linear ordering
find :: Ord a => a -> Tree a -> Bool
find _ Empty  =  False
find x (Node a l r)
  | x < a  =  find x l
  | a < x  =  find x r
  | otherwise  =  True
```

```
insert :: Ord a => a -> Tree a -> Tree a
insert x Empty  =  Node x Empty Empty
insert x (Node a l r)
  | x < a  =  Node a (insert x l) r
  | a < x  =  Node a l (insert x r)
  | otherwise  =  Node a l r
```

Example

```
insert 6 (Node 5 Empty (Node 7 Empty Empty))
= Node 5 Empty (insert 6 (Node 7 Empty Empty))
= Node 5 Empty (Node 7 (insert 6 Empty) Empty)
= Node 5 Empty (Node 7 (Node 6 Empty Empty) Empty)
```

QuickCheck for Tree

```
import Control.Monad
import Test.QuickCheck

-- for QuickCheck: test data generator for Trees
instance Arbitrary a => Arbitrary (Tree a) where
  arbitrary = sized tree
    where
    tree 0 = return Empty
    tree n | n > 0 =
      oneof [return Empty,
             liftM3 Node arbitrary (tree (n ‘div‘ 2))
                                   (tree (n ‘div‘ 2))]
```

```
prop_find_insert :: Int -> Int -> Tree Int -> Bool
prop_find_insert x y t =
  find x (insert y t) == ???
```

(`Int` not optimal for QuickCheck)

Edit distance (see Thompson)

Problem: how to get from one word to another, with a *minimal* number of “edits”.

Example: from `"fish"` to `"chips"`

Applications: DNA Analysis, Unix `diff` command

```
data Edit = Change Char
          | Copy
          | Delete
          | Insert Char
          deriving (Eq, Show)

transform :: String -> String -> [Edit]

transform [] ys  =  map Insert ys
transform xs []  =  replicate (length xs) Delete
transform (x:xs) (y:ys)
  | x == y       =  Copy : transform xs ys
  | otherwise    =  best [Change y : transform xs ys,
                          Delete   : transform xs (y:ys),
                          Insert y : transform (x:xs) ys]
```

```
best :: [[Edit]] -> [Edit]
best [x]   = x
best (x:xs)
  | cost x <= cost b    = x
  | otherwise           = b
  where b = best xs

cost :: [Edit] -> Int
cost = length . filter (/=Copy)
```

Example: What is the edit distance
from `"trittin"` to `"tarantino"`?

`transform "trittin" "tarantino" = ?`

Complexity of `transform`: time $O(\quad\quad)$

The edit distance problem can be solved in time $O(mn)$
with *dynamic programming*

8.2 The general case

$$\begin{array}{l} \texttt{data}\ T\ a_1\ \ldots\ a_p\ \texttt{=} \\ \quad C_1\ t_{11}\ \ldots\ t_{1k_1}\ \texttt{|} \\ \quad \vdots \\ \quad C_n\ t_{n1}\ \ldots\ t_{nk_n} \end{array}$$

defines the *constructors*

$$\begin{array}{l} C_1\ \texttt{::}\ t_{11}\ \texttt{->}\ \ldots\ t_{1k_1}\ \texttt{->}\ T\ a_1\ \ldots\ a_p \\ \vdots \\ C_n\ \texttt{::}\ t_{n1}\ \texttt{->}\ \ldots\ t_{nk_n}\ \texttt{->}\ T\ a_1\ \ldots\ a_p \end{array}$$

Patterns revisited

Patterns are expressions that consist only of constructors and variables (which must not occur twice):
A *pattern* can be

- a variable (incl. `_`)
- a literal like `1`, `'a'`, `"xyz"`, ...
- a tuple `(`p_1`, ..., `p_n`)` where each p_i is a pattern
- a constructor pattern C p_1 `...` p_n where
  C is a `data` constructor (incl. `True`, `False`, `[]` and `(:)`)
  and each p_i is a pattern

8.3 Case study: boolean formulas

```
type Name = String

data Form = F | T
          | Var Name
          | Not Form
          | And Form Form
          | Or Form Form
          deriving Eq
```

Example: `Or (Var "p") (Not(Var "p"))`

More readable: symbolic infix constructors, must start with `:`

```
data Form = F | T | Var Name
          | Not Form
          | Form :&: Form
          | Form :|: Form
          deriving Eq
```

Now: `Var "p" :|: Not(Var "p")`

Pretty printing

```
par :: String -> String
par s = "(" ++ s ++ ")"

instance Show Form where
  show F = "F"
  show T = "T"
  show (Var x) = x
  show (Not p) = par("~" ++ show p)
  show (p :&: q) = par(show p ++ " & " ++ show q)
  show (p :|: q) = par(show p ++ " | " ++ show q)
```

```
> Var "p" :&: Not(Var "p")
(p & (~p))
```

Syntax versus meaning

`Form` is the *syntax* of boolean formulas, not their meaning:

`Not(Not T)` and `T` mean the same but are different:

```
Not(Not T) /= T
```

What is the meaning of a `Form`?

Its value!?

But what is the value of `Var "p"` ?

```
-- Wertebelegung
type Valuation = [(Name,Bool)]

eval :: Valuation -> Form -> Bool
eval _ F = False
eval _ T = True
eval v (Var x) = the(lookup x v)  where  the(Just b) = b
eval v (Not p) = not(eval v p)
eval v (p :&: q) = eval v p && eval v q
eval v (p :|: q) = eval v p || eval v q
```

```
> eval [("a",False), ("b",False)]
   (Not(Var "a") :&: Not(Var "b"))
True
```

All valuations for a given list of variable names:

```
vals :: [Name] -> [Valuation]
vals [] = [[]]
vals (x:xs) = [ (x,False):v | v <- vals xs ] ++
              [ (x,True):v  | v <- vals xs ]

vals ["b"]
= [("b",False):v | v <- vals [[]]] ++
  [("b",True):v  | v <- vals [[]]]
= [("b",False):[]] ++ [("b",True):[]]
= [("b",False), ("b",True)]

vals ["a","b"]
= [("a",False):v | v <- vals ["b"]] ++
  [("a",True):v  | v <- vals ["b"]]
= [[("a",False),("b",False)] ++ [("a",False),("b",True)] ++
  [[("a",True), ("b",False)] ++ [("a",True), ("b",True)]
```

Does `vals` construct *all* valuations?

```
prop_vals1 xs =
  length(vals xs) ==  2 ^ length xs

prop_vals2 xs =
  distinct (vals xs)

distinct :: Eq a => [a] -> Bool
distinct [] = True
distinct (x:xs) = not(elem x xs) && distinct xs
```

Demo

Restrict size of test cases:

```
prop_vals1' xs =
  length xs <= 10 ==>
  length(vals xs) == 2 ^ length xs

prop_vals2' xs =
  length xs <= 10 ==> distinct (vals xs)
```

Demo

Satisfiable and tautology

```
satisfiable :: Form -> Bool
satisfiable p = or [eval v p | v <- vals(vars p)]

tautology :: Form -> Bool
tautology = not . satisfiable . Not

vars :: Form -> [Name]
vars F = []
vars T = []
vars (Var x) = [x]
vars (Not p) = vars p
vars (p :&: q) = nub (vars p ++ vars q)
vars (p :|: q) = nub (vars p ++ vars q)
```

```
p0 :: Form
p0 =  (Var "a" :&: Var "b") :|:
      (Not (Var "a") :&: Not (Var "b"))
```

```
> vals (vars p0)
[[("a",False),("b",False)], [("a",False),("b",True)],
 [("a",True), ("b",False)], [("a",True), ("b",True )]]

> [ eval v p0 | v <- vals (vars p0) ]
[True, False, False, True]

> satisfiable p0
True
```

Simplifying a formula: `Not` inside?

```
isSimple :: Form -> Bool
isSimple (Not p)     =  not (isOp p)
  where
  isOp (Not p)     =  True
  isOp (p :&: q)   =  True
  isOp (p :|: q)   =  True
  isOp p           =  False
isSimple (p :&: q)  =  isSimple p && isSimple q
isSimple (p :|: q)  =  isSimple p && isSimple q
isSimple p          =  True
```

Simplifying a formula: `Not` inside!

```
simplify :: Form -> Form
simplify (Not p)    =  pushNot (simplify p)
  where
  pushNot (Not p)    =  p
  pushNot (p :&: q)  =  pushNot p :|: pushNot q
  pushNot (p :|: q)  =  pushNot p :&: pushNot q
  pushNot p          =  Not p
simplify (p :&: q)  =  simplify q :&: simplify q
simplify (p :|: q)  =  simplify p :|: simplify q
simplify p          =  p
```

Quickcheck

```
-- for QuickCheck: test data generator for Form
instance Arbitrary Form where
  arbitrary = sized prop
    where
    prop 0  =
      oneof [return F,
             return T,
             liftM Var arbitrary]
    prop n | n > 0 =
      oneof
        [return F,
         return T,
         liftM Var arbitrary,
         liftM Not (prop (n-1)),
         liftM2 (:&:) (prop(n `div` 2)) (prop(n `div` 2)),
         liftM2 (:|:) (prop(n `div` 2)) (prop(n `div` 2))]
```

```
prop_simplify p  =  isSimple(simplify p)
```

8.4 Structural induction

Structural induction for `Tree`

```
data Tree a = Empty | Node a (Tree a) (Tree a)
```

To prove property `P(t)` for all finite `t :: Tree a`

Base case: Prove `P(Empty)` and

Induction step: Prove `P(Node x t1 t2)`
assuming the induction hypotheses `P(t1)` and `P(t2)`.
(`x`, `t1` and `t2` are new variables)

Example

```
flat :: Tree a -> [a]
flat Empty  =  []
flat (Node x t1 t2)  =
  flat t1 ++ [x] ++ flat t2

mapTree :: (a -> b) -> Tree a -> Tree b
mapTree f Empty  =  Empty
mapTree f (Node x t1 t2)  =
  Node (f x) (mapTree f t1) (mapTree f t2)
```

Lemma `flat (mapTree f t) = map f (flat t)`

Proof by structural induction on `t`

Induction step:

IH1: `flat (mapTree f t1) = map f (flat t1)`
IH2: `flat (mapTree f t2) = map f (flat t2)`

To show: `flat (mapTree f (Node x t1 t2)) =`
`map f (flat (Node x t1 t2))`

```
flat (mapTree f (Node x t1 t2))
= flat (Node (f x) (mapTree f t1) (mapTree f t2))
= flat (mapTree f t1) ++ [f x] ++ flat (mapTree f t2)
= map f (flat t1) ++ [f x] ++ map f (flat t2)
    -- by IH1 and IH2
```

```
map f (flat (Node x t1 t2))
= map f (flat t1 ++ [x] ++ flat t2)
= map f (flat t1) ++ [f x] ++ map f (flat t2)
    -- by lemma distributivity of map over ++
```

Note: Base case and `-- by def of ...` omitted

The general (regular) case

`data T a = ...`

Assumption: `T` is a *regular* data type:

Each constructor C_i of `T` must have a type

t_1 `->` `...` `->` t_{n_i} `-> T a`

such that each t_j is either `T a` or does not contain `T`

To prove property $P(t)$ for all finite t `:: T a`:

prove for each constructor C_i that $P(C_i\ x_1 \ldots x_{n_i})$

assuming the induction hypotheses $P(x_j)$ for all j s.t. $t_j =$ `T a`

Example of non-regular type: `data T = C [T]`

9. Modules and Abstract Data Types

- Modules
- Abstract Data Types
- Correctness

9.1 Modules

Module = collection of type, function, class etc definitions

Purposes:

- Grouping
- Interfaces
- Division of labour
- Name space management: `M.f` vs `f`
- Information hiding

GHC: one module per file

Recommendation: module `M` in file `M.hs`

Module header

```
module M where   -- M must start with capital letter
↑
```

All definitions must start in this column

- Exports everything defined in `M` (at the top level)

Selective export:

```
module M (T, f, ...)  where
```

- Exports only `T, f, ...`

Exporting data types

```
module M (T) where
data T = ...
```

- Exports only `T`, but not its constructors

```
module M (T(C,D,...)) where
data T = ...
```

- Exports `T` and its constructors `C`, `D`, . . .

```
module M (T(..)) where
data T = ...
```

- Exports `T` and all of its constructors

Not permitted: `module M (T,C,D) where` (why?)

Exporting modules

By default, modules do not export names from imported modules

```
module B where
import A
...
```

```
module A where
f = ...
...
```

⟹ B does not export `f`

Unless the names are mentioned in the export list

```
module B (f) where
import A
...
```

Or the whole module is exported

```
module B (module A) where
import A
...
```

`import`

By default, everything that is exported is imported

```
module B where
import A
...
```

```
module A where
f = ...
g = ...
```

⟹ B imports `f` and `g`

Unless an import list is specified

```
module B where
import A (f)
...
```

⟹ B imports only `f`

Or specific names are hidden

```
module B where
import A hiding (g)
...
```

`qualified`

```
import A
import B
import C
... f ...
```

Where does `f` come from??

Clearer: *qualified names*

```
... A.f ...
```

Can be enforced:

```
import qualified A
```

⟹ must always write `A.f`

Renaming modules

```
import TotallyAwesomeModule

... TotallyAwesomeModule.f ...
```

Painful

More readable:

```
import qualified TotallyAwesomeModule as TAM

... TAM.f ...
```

For the full description of the module system
see the Haskell report

9.2 Abstract Data Types

Abstract Data Types do not expose their internal representation

Why? Example: sets implemented as lists without duplicates

- Could create illegal value: `[1, 1]`
- Could distinguish what should be indistinguishable:
  `[1, 2] /= [2, 1]`
- Cannot easily change representation later

Example: Sets

```
module Set where
-- sets are represented as lists w/o duplicates
type Set a = [a]
empty  :: Set a
empty  =  []
insert :: a -> Set a -> Set a
insert x xs  =  ...
isin :: a -> Set a -> Set a
isin x xs  =  ...
size :: Set a -> Integer
size xs  =  ...
```

Exposes everything

Allows nonsense like `Set.size [1,1]`

Better

```
module Set (Set, empty, insert, isin, size) where
-- Interface
empty  :: Set a
insert :: Eq a => a -> Set a -> Set a
isin   :: Eq a => a -> Set a -> Bool
size   :: Set a -> Int
-- Implementation
type Set a = [a]
...
```

- Explicit export list/interface
- But representation still not hidden
  Does not help: hiding the type name `Set`

Hiding the representation

```
module Set (Set, empty, insert, isin, size) where
-- Interface
...
-- Implementation
data Set a = S [a]

empty = S []
insert x (S xs) = S(if elem x xs then xs else x:xs)
isin x (S xs) = elem x xs
size (S xs) = length xs
```

Cannot construct values of type `Set` outside of module `Set` because `S` is not exported

```
Test.hs:3:11:  Not in scope:  data constructor ‘S’
```

Uniform naming convention: $S \rightsquigarrow$ Set

```
module Set (Set, empty, insert, isin, size) where
-- Interface
...
-- Implementation
data Set a = Set [a]

empty = Set []
insert x (Set xs) = Set(if elem x xs then xs else x:xs)
isin x (Set xs) = elem x xs
size (Set xs) = length xs
```

Which `Set` is exported?

Slightly more efficient: `newtype`

```
module Set (Set, empty, insert, isin, size) where
-- Interface
...
-- Implementation
newtype Set a = Set [a]

empty = Set []
insert x (Set xs) = Set(if elem x xs then xs else x:xs)
isin x (Set xs) = elem x xs
size (Set xs) = length xs
```

Conceptual insight

Data representation can be hidden
by wrapping data up in a constructor that is not exported

What if `Set` is already a data type?

```
module SetByTree (Set, empty, insert, isin, size) where

-- Interface
empty  :: Set a
insert :: Ord a => a -> Set a -> Set a
isin   :: Ord a => a -> Set a -> Bool
size   :: Set a -> Integer

-- Implementation
type Set a = Tree a
data Tree a = Empty | Node a (Tree a) (Tree a)
```

No need for `newtype`:
The representation of `Tree` is hidden
as long as its constructors are hidden

Beware of ==

```
module SetByTree (Set, empty, insert, isin, size) where
...
type Set a = Tree a
data Tree a = Empty | Node a (Tree a) (Tree a)
              deriving (Eq)
...
```

Class instances are automatically exported and cannot be hidden

Client module:

```
import SetByTree
... insert 2 (insert 1 empty) ==
    insert 1 (insert 2 empty)
...
```

Result is probably `False` — representation is partly exposed!

The proper treatment of ==

Some alternatives:

- Do not make `Tree` an instance of `Eq`
- Hide representation:

```
-- do not export constructor Set:
newtype Set a = Set (Tree a)
data Tree a = Empty | Node a (Tree a) (Tree a)
              deriving (Eq)
```

- Define the right == on `Tree`:

```
instance Eq a => Eq(Tree a) where
  t1 == t2  =  elems t1 == elems t2
    where
    elems Empty = []
    elems (Node x t1 t2) = elems t1 ++ [x] ++ elems t2
```

Similar for all class instances,
not just `Eq`

9.3 Correctness

Why is module `Set` a correct implementation of (finite) sets?

Because `empty` simulates $\{\}$
and `insert _ _` simulates $\{\_\} \cup \_$
and `isin _ _` simulates $\_ \in \_$
and `size _` simulates $|\_|$

Each concrete operation on the implementation type of lists
simulates its abstract counterpart on sets

NB: We relate Haskell to mathematics

For uniformity we write $\{a\}$ for the type of finite sets over type `a`

From lists to sets

Each list $[x_1, \dots, x_n]$ represents the set $\{x_1, \dots, x_n\}$.

Abstraction function α `:: [a] ->` $\{$`a`$\}$

$$\alpha[x_1, \dots, x_n] = \{x_1, \dots, x_n\}$$

In Haskell style:

$$\alpha\ \texttt{[]}\ \texttt{=}\ \{\}$$
$$\alpha\ \texttt{(x:xs)}\ \texttt{=}\ \{\texttt{x}\} \cup \alpha\ \texttt{xs}$$

What does it mean that "lists simulate (implement) sets":

$$\alpha\ \text{(concrete operation)} = \text{abstract operation}$$
$$\alpha\ \texttt{empty} = \{\}$$
$$\alpha\ \texttt{(insert x xs)} = \{\texttt{x}\} \cup \alpha\ \texttt{xs}$$
$$\texttt{isin x xs} = \texttt{x} \in \alpha\ \texttt{xs}$$
$$\texttt{size xs} = |\alpha\ \texttt{xs}|$$

For the mathematically enclined:
α must be a homomorphism

Implementation I: lists with duplicates

```
empty        =  []
insert x xs  =  x : xs
isin x xs    =  elem x xs
size xs      =  length(nub xs)
```

The simulation requirements:

$$
\begin{aligned}
\alpha\ \texttt{empty} &= \{\} \\
\alpha\ (\texttt{insert x xs}) &= \{\texttt{x}\} \cup \alpha\ \texttt{xs} \\
\texttt{isin x xs} &= \texttt{x} \in \alpha\ \texttt{xs} \\
\texttt{size xs} &= |\alpha\ \texttt{xs}|
\end{aligned}
$$

Two proofs immediate, two need lemmas proved by induction

Implementation II: lists without duplicates

```
empty         =  []
insert x xs   =  if elem x xs then xs else x:xs
isin x xs     =  elem x xs
size xs       =  length xs
```

The simulation requirements:

$$\begin{aligned}\alpha\ \texttt{empty} &= \{\}\\ \alpha\ (\texttt{insert x xs}) &= \{\texttt{x}\} \cup \alpha\ \texttt{xs}\\ \texttt{isin x xs} &= \texttt{x} \in \alpha\ \texttt{xs}\\ \texttt{size xs} &= |\alpha\ \texttt{xs}|\end{aligned}$$

Needs *invariant* that `xs` contains no duplicates

```
invar :: [a] -> Bool
invar []        =  True
invar (x:xs)    =  not(elem x xs) && invar xs
```

Implementation II: lists without duplicates

```
empty         =  []
insert x xs   =  if elem x xs then xs else x:xs
isin x xs     =  elem x xs
size xs       =  length xs
```

Revised simulation requirements:

$$
\begin{array}{rrcl}
 & \alpha\ \texttt{empty} & = & \{\} \\
\texttt{invar xs} \Longrightarrow & \alpha\ (\texttt{insert x xs}) & = & \{\texttt{x}\} \cup \alpha\ \texttt{xs} \\
\texttt{invar xs} \Longrightarrow & \texttt{isin x xs} & = & \texttt{x} \in \alpha\ \texttt{xs} \\
\texttt{invar xs} \Longrightarrow & \texttt{size xs} & = & |\alpha\ \texttt{xs}|
\end{array}
$$

Proofs omitted. Anything else?

`invar` must be invariant!

In an imperative context:

If `invar` is true before an operation,
it must also be true after the operation

In a functional context:

If `invar` is true for the arguments of an operation,
it must also be true for the result of the operation

`invar` is *preserved* by every operation

$$\begin{array}{rl} & \texttt{invar empty} \\ \texttt{invar xs} \Longrightarrow & \texttt{invar (insert x xs)} \end{array}$$

Proofs do not even need induction

Summary

Let C and A be two modules that have the same interface:

a type T and a set of functions F

To prove that C is a correct implementation of A define

an *abstraction function* α :: $C.T$ -> $A.T$

and an *invariant* `invar` :: $C.T$ -> `Bool`

and prove for each $f \in F$:

- `invar` is invariant:

$$\texttt{invar}\ x_1 \wedge \cdots \wedge \texttt{invar}\ x_n \implies \texttt{invar}\ (C.f\ x_1\ \ldots\ x_n)$$

(where `invar` is `True` on types other than $C.T$)

- $C.f$ simulates $A.f$:

$$\texttt{invar}\ x_1 \wedge \cdots \wedge \texttt{invar}\ x_n \implies$$
$$\alpha(C.f\ x_1\ \ldots\ x_n) = A.f\ (\alpha\ x_1)\ \ldots\ (\alpha\ x_n)$$

(where α is the identity on types other than $C.T$)

10. Case Study: Huffman Coding

See Thompson, blackboard and the source files on the web page

11. Case Study: Parsing

See blackboard and the source files on the web page

12. Lazy evaluation

Applications of lazy evaluation
Infinite lists

Introduction

So far, we have not looked at the details of how Haskell expressions are evaluated. The evaluation strategy is called

lazy evaluation („verzögerte Auswertung")

Advantages:

- Avoids unnecessary evaluations
- Terminates as often as possible
- Supports infinite lists
- Increases modularity

Therefore Haskell is called a *lazy functional langauge*.
Haskell is the only mainstream lazy functional language.

Evaluating expressions

Expressions are evaluated (*reduced*) by successively applying definitions until no further reduction is possible.

Example:

```
sq :: Integer -> Integer
sq n  =  n * n
```

One evaluation:

$$\texttt{sq(}\underline{\texttt{3+4}}\texttt{)} \;=\; \underline{\texttt{sq}}\ \texttt{7} \;=\; \underline{\texttt{7 * 7}} \;=\; \texttt{49}$$

Another evaluation:

$$\underline{\texttt{sq}}\texttt{(3+4)} \;=\; \underline{\texttt{(3+4)}}\ \texttt{* (3+4)} \;=\; \texttt{7 *}\ \underline{\texttt{(3+4)}} \;=\; \underline{\texttt{7 * 7}} \;=\; \texttt{49}$$

Theorem

Any two terminating evaluations of the same Haskell expression lead to the same final result.

This is not the case in languages with side effects:

Example

Let `n` have value 0 initially.

Two evaluations:

$$\underline{\texttt{n}}\ \texttt{+ (n := 1)} \quad = \quad \texttt{0 + (}\underline{\texttt{n := 1}}\texttt{)} \quad = \quad \underline{\texttt{0 + 1}} \quad = \quad \texttt{1}$$

$$\texttt{n + (}\underline{\texttt{n := 1}}\texttt{)} \quad = \quad \underline{\texttt{n}}\ \texttt{+ 1} \quad = \quad \underline{\texttt{1 + 1}} \quad = \quad \texttt{2}$$

Reduction strategies

An expression may have many reducible subexpressions:

$$\underline{\texttt{sq (}\underline{\texttt{3+4}}\texttt{)}}$$

Terminology: *redex* = **red**ucible **ex**pression

Two common reduction strategies:

Innermost reduction Always reduce an innermost redex.
Corresponds to *call by value*:
Arguments are evaluated
before they are substituted into the function body
`sq (3+4)  =  sq 7  =  7 * 7`

Outermost reduction Always reduce an outermost redex.
Corresponds to *call by name:*
The unevaluated arguments
are substituted into the the function body
`sq (3+4)  =  (3+4) * (3+4)`

Comparison: termination

Definition:

```
loop = tail loop
```

Innermost reduction:

```
fst (1,loop)  =  fst(1,tail loop)
              =  fst(1,tail(tail loop))
              =  ...
```

Outermost reduction:

```
fst (1,loop)  =  1
```

Theorem If expression `e` has a terminating reduction sequence, then outermost reduction of `e` also terminates.

Outermost reduction terminates as often as possible

Why is this useful?

Example

Can build your own control constructs:

```
switch :: Int -> a -> a -> a
switch n x y
  | n > 0       =  x
  | otherwise   =  y

fac :: Int -> Int
fac n = switch n (n * fac(n-1)) 1
```

Comparison: Number of steps

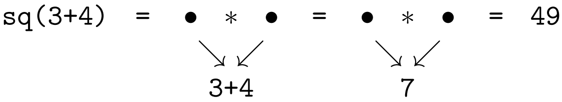
Innermost reduction:

```
sq (3+4) =  sq 7  =  7 * 7  =  49
```

Outermost reduction:

```
sq(3+4)  =  (3+4)*(3+4)  =  7*(3+4)  =  7*7  =  49
```

More outermost than innermost steps!
How can outermost reduction be improved?
Sharing!



`sq(3+4)` = • $*$ • = • $*$ • = `49`

`3+4` `7`

The expression 3+4 is only evaluated *once!*

Lazy evaluation := outermost reduction + sharing

Theorem
Lazy evaluation never needs more steps than innermost reduction.

The principles of lazy evaluation:

- Arguments of functions are evaluated only if needed to continue the evaluation of the function.
- Arguments are not necessarily evaluated fully, but only far enough to evaluate the function. (Remember `fst (1,loop)`)
- Each argument is evaluated at most once (sharing!)

Pattern matching

Example

```
f :: [Int] -> [Int] -> Int
f []     ys     =  0
f (x:xs) []     =  0
f (x:xs) (y:ys) =  x+y
```

Lazy evaluation:

```
f [1..3] [7..9]                    -- does f.1 match?
=  f (1 : [2..3]) [7..9]           -- does f.2 match?
=  f (1 : [2..3]) (7 : [8..9])  -- does f.3 match?
=  1+7
=  8
```

Guards

Example

```
f m n p | m >= n && m >= p  =  m
        | n >= m && n >= p  =  n
        | otherwise         =  p
```

Lazy evaluation:

```
f (2+3) (4-1) (3+9)
  ?  2+3 >= 4-1 && 2+3 >= 3+9
  ?  =  5 >= 3 && 5 >= 3+9
  ?  =  True   && 5 >= 3+9
  ?  =  5 >= 3+9
  ?  =  5 >= 12
  ?  =  False
  ?  3 >= 5 && 3 >= 12
  ?  =  False && 3 >= 12
  ?  =  False
  ?  otherwise = True
=  12
```

`where`

Same principle: definitions in `where` clauses are only evaluated when needed and only as much as needed.

Lambda

Haskell never reduces inside a lambda

Example: `\x -> False && x` cannot be reduced

Reasons:

- Functions are black boxes
- All you can do with a function is apply it

Example:

```
(\x -> False && x) True  =  False && True  =  False
```

Predefined functions

They behave like their Haskell definition (if they have one):

```
(&&) :: Bool -> Bool -> Bool
True  && y  =  y
False && y  =  False
```

Or they evaluate their arguments first: basic arithmetic

Slogan

Lazy evaluation evaluates an expression only when needed
and only as much as needed.

(*"Call by need"*)

12.1 Applications of lazy evaluation

The minimum of a list

```
min   =  head . inSort

inSort :: Ord a => [a] -> [a]
inSort []       =  []
inSort (x:xs)   =  ins x (inSort xs)

ins :: Ord a => a -> [a] -> [a]
ins x []   =  [x]
ins x (y:ys) | x <= y      =  x : y : ys
             | otherwise   =  y : ins x ys
```

```
⟹  inSort [6,1,7,5]
   =  ins 6 (ins 1 (ins 7 (ins 5 [])))
```

```
min [6,1,7,5]  =  head(inSort [6,1,7,5])
=  head(ins 6 (ins 1 (ins 7 (ins 5 []))))
=  head(ins 6 (ins 1 (ins 7 (5 : []))))
=  head(ins 6 (ins 1 (5 : ins 7 [])))
=  head(ins 6 (1 : 5 : ins 7 []))
=  head(1 : ins 6 (5 : ins 7 []))
=  1
```

Lazy evaluation needs only linear time
although `inSort` is quadratic
because the sorted list is never constructed completely

Warning: this depends on the exact algorithm and does not work so nicely with all sorting functions!

Parser: all results

```
type Parser a b = [a] -> Maybe (b, [a])
```

$\rightsquigarrow$

```
type Parser a b = [a] -> [ (b, [a]) ]
```

```
p1 ||| p2  =  \as -> case p1 as of
                        Nothing -> p2 as
                        just -> just
```

⇝

```
p1 ||| p2  =  \xs -> p1 xs ++ p2 xs
```

```
p1 *** p2  =  \xs ->
  case p1 xs of
    Nothing -> Nothing
    Just(b,ys) -> case p2 ys of
                    Nothing -> Nothing
                    Just(c,zs) -> Just((b,c),zs)
```

⇝

```
p1 *** p2  =  \xs ->
  [ ((b,c),zs) | (b,ys) <- p1 xs, (c,zs) <- p2 ys]
```

```
p >>> f  =  \xs ->
  case p xs of
    Nothing -> Nothing
    Just(b, ys) -> Just(f b, ys)
```

⇝

```
p >>> f  =  [(f b,ys) | (b,ys) <- p xs]
```

12.2 Infinite lists

Example

A recursive definition

```
ones :: [Int]
ones  =  1 : ones
```

that defines an infinite list of 1s:

```
ones  =  1 : ones  =  1 : 1 : ones  =  ...
```

What GHCi has to say about it:

```
> ones
[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
```

Haskell lists can be finite or infinite

Printing an infinite list does not terminate

But Haskell can compute with infinite lists, thanks to lazy evaluation:

```
> head ones
1
```

Remember:

> Lazy evaluation evaluates an expression only as much as needed

Outermost reduction: `head ones   =   head (1 : ones)   =   1`

Innermost reduction:

```
head ones
=  head (1 : ones)
=  head (1 : 1 : ones)
=  ...
```

Haskell lists are never actually infinite but only potentially infinite

Lazy evaluation computes as much of the infinite list as needed

This is how partially evaluated lists are represented internally:

`1 : 2 : 3 :` $\boxed{\text{code pointer to compute rest}}$

In general: finite prefix followed by code pointer

Why (potentially) infinite lists?

- They come for free with lazy evaluation
- They increase modularity:
  list producer does not need to know
  how much of the list the consumer wants

Example: The sieve of Eratosthenes

1. Create the list 2, 3, 4, . . .
2. Output the first value p in the list as a prime.
3. Delete all multiples of p from the list
4. Goto step 2

2 3 4 5 6 7 8 9 10 11 12 . . .
2 3 5 7 11 . . .

In Haskell:

```haskell
primes :: [Int]
primes  =  sieve [2..]

sieve :: [Int] -> [Int]
sieve (p:xs)  =  p : sieve [x | x <- xs, x `mod` p /= 0]
```

Lazy evaluation:

```
primes  =  sieve [2..]  =  sieve (2:[3..])
=  2 : sieve [x | x <- [3..], x `mod` 2 /= 0]
=  2 : sieve [x | x <- 3:[4..],  x `mod` 2 /= 0]
=  2 : sieve (3 : [x | x <- [4..],  x `mod` 2 /= 0])
=  2 : 3 : sieve [x | x <- [x|x <- [4..],  x `mod` 2 /= 0]
                      x `mod` 3 /= 0]
= ...
```

Modularity!

The first 10 primes:

```
> take 10 primes
[2,3,5,7,11,13,17,19,23,29]
```

The primes between 100 and 150:

```
> takeWhile (<150) (dropWhile (<100) primes)
[101,103,107,109,113,127,131,137,139,149]
```

All twin primes:

```
> [(p,q) | (p,q) <- zip primes (tail primes), p+2==q]
[(3,5),(5,7),(11,13),(17,19),(29,31),(41,43),(59,61),(71,7
```

Primality test?

```
> 101 `elem` primes
True
```

```
> 102 `elem` primes
```

nontermination

```
prime n  =  n == head (dropWhile (<n) primes)
```

13. I/O and Monads

- I/O
- Monads

13.1 I/O

- So far, only batch programs:
  given the full input at the beginning,
  the full output is produced at the end
- Now, interactive programs:
  read input and write output
  while the program is running

The problem

- Haskell programs are pure mathematical functions:

  Haskell programs have no side effects

- Readind and writing are side effects:

  Interactive programs have side effects

An impure solution

Most languages allow functions to perform I/O
without reflecting it in their type.

Assume that Haskell were to provide an input function

```
inputInt :: Int
```

Now all functions are potentially perform side effects.

Now we can no longer reason about Haskell like in mathematics:

```
inputInt - inputInt   =   0
inputInt + inputInt   =   2*inputInt
                      ...
```

are no longer true.

The pure solution

Haskell distinguishes expressions without side effects
from expressions with side effects (*actions*) by their type:

`IO a`

is the type of (I/O) actions that return a value of type `a`.

Example

`Char`: the type of pure expressions that return a `Char`
`IO Char`: the type of actions that return an `Char`
`IO ()`: the type of actions that return no result value

`()`

- Type `()` is the type of empty tuples (no fields).
- The only value of type `()` is `()`, the empty tuple.
- Therefore `IO ()` is the type of actions
  that return the dummy value `()`
  (because every action must return some value)

Basic actions

- `getChar :: IO Char`

  Reads a `Char` from standard input,
  echoes it to standard output,
  and returns it as the result

- `putChar :: Char -> IO ()`

  Writes a `Char` to standard output,
  and returns no result

- `return :: a -> IO a`

  Performs no action,
  just returns the given value as a result

Sequencing: `do`

A sequence of actions can be combined into a single action with the keyword `do`

Example

```
get2 :: IO (Char,Char)
get2 = do x <- getChar   -- result is named x
          getChar        -- result is ignored
          y <- getChar
          return (x,y)
```

General format (observe layout!):

$$\begin{array}{ll}\texttt{do} & a_1 \\ & \vdots \\ & a_n\end{array}$$

where each a_i can be one of

- an action
  Effect: execute action
- x `<-` *action*
  Effect: execute *action*, give result the name x
- `let` x `=` *expr*
  Effect: give *expr* the name x
  Lazy: *expr* is only evaluated when x is needed!

Derived primitives

Write a string to standard output:

```
putStr :: String -> IO ()
putStr []       =  return ()
putStr (c:cs)   =  do putChar c
                      putStr cs
```

Write a line to standard output:

```
putStrLn :: IO ()
putStrLn cs  =  putStr (cs ++ '\n')
```

Read a line from standard input:

```
getLine :: IO String
getLine = do x <- getChar
             if x == '\n' then
                return []
             else
                do xs <- getLine
                   return (x:xs)
```

Actions are normal Haskell values and can be combined as usual, for example with `if-then-else`.

Example

Prompt for a string and display its length:

```
strLen :: IO ()
strLen  =  do putStr "Enter a string: "
              xs <- getLine
              putStr "The string has "
              putStr (show (length xs))
              putStrLn " characters"
```

```
> strLen

Enter a string: abc
The string has 3 characters
```

How to read other types

Input string and convert

Useful class:

```
class Read a where
  read :: String -> a
```

Most predefined types are in class `Read`.

Example:

```
getInt :: IO Integer
getInt = do xs <- getLine
            return (read xs)
```

File I/O

So far implicit: read from `stdin :: Handle`
write to `stdout :: Handle`

`data Handle`

> *Haskell defines operations to read and write characters from and to files, represented by values of type* `Handle`*. Each value of this type is a handle: a record used by the Haskell run-time system to manage I/O with file system objects.*

Details: Haskell IO library

Case study

The game of Hangman
in file `Hang.hs`

Once `IO`, always `IO`

You cannot add I/O to a function without "polluting" its type

For example

```
sq :: Int -> Int          cube :: Int -> Int
sq x = x*x                cube x = x * sq x
```

Let us try to make `sq` print out some message:

```
sq x = do putStr("I am in sq!")
          return(x*x)
```

What is the type of `sq` now? `Int -> IO Int`
And this is what happens to `cube`:

```
cube x = do x2 <- sq x
            return(x * x2)
```

Haskell is a pure functional language
Functions that have side effects must show this in their type
I/O is a side effect

13.2 Monads

>>= ('bind'), or what do really means

Primitive:

```
(>>=) :: IO a -> (a -> IO b) -> IO b
```

How it works:

act `>>=` *f*	execute action *act* `:: IO a`
	which returns a result *v* `:: a`
	then evaluate *f* *v*
	which returns a result of type `IO b`

`do` *x* `<-` act_1
    act_2

is syntax for

act_1 `>>= (\x ->` act_2`)`

Example

```
do x <- getChar
   putChar x
```

⇝

```
getChar >>= (\x -> putChar x)
```

In general

`do` x_1 `<-` a_1
$\vdots$
x_n `<-` a_n
act

is syntax for

a_1 `>>=` `\`x_1 `->`
$\vdots$
a_n `>>=` `\`x_n `->`
act

Beyond `IO`: *Monads*

```
class Monad m where
  (>>=) :: m a -> (a -> m b) -> m b
  return :: a -> m a
```

- `m` is a type constructor
- `do` notation is defined for every monad

Only example of monad so far: `IO`
Let's examine some more.

Maybe as a monad

A frequent code pattern when working with Maybe:

```
case m of
  Nothing -> Nothing
  Just x -> ...
```

This pattern can be hidden inside >>=:

```
instance Monad Maybe where
  m >>= f  =  case m of
                Nothing -> Nothing
                Just x -> f x
  return v  =   Just v
```

Failure (= `Nothing`) propagation and unwrapping of `Just` is now built into `do`!

```
instance Monad Maybe where
  m >>= f  =  case m of
                Nothing -> Nothing
                Just x -> f x
  return v  =  Just v
```

Example: evaluation of `Form`

```
eval :: [(Name,Bool)] -> Form -> Maybe Bool
eval _ T  =  return True
eval _ F  =  return False
eval v (Var x)  =  lookup x v
eval v (f1 :&: f2)  =  do b1 <- eval v f1
                          b2 <- eval v f2
                          return (b1 && b2)
...
```

Example:

```
p1 *** p2  =  \xs ->
  case p1 xs of
    Nothing -> Nothing
    Just(b,ys) -> case p2 ys of
                    Nothing -> Nothing
                    Just(c,zs) -> Just((b,c),zs)
```

$\rightsquigarrow$

```
p1 *** p2  = \xs ->
  do (b,ys) <- p1 xs
     (c,zs) <- p2 ys
     return ((b,c),zs)
```

The `do` version has a much more general type `Monad m => ...`

`Maybe` models possible failure with `Just`/`Nothing`

The `do` of the `Maybe` monad hides `Just`/`Nothing`
and propagates failure automatically

List as a monad

```
instance Monad [] where
  xs >>= f  =  concat(map f xs)
  return v  =  [v]
```

Now we can compose computations on list nicely (via `do`).

Example

```
dfs :: (a -> [a]) -> (a -> Bool) -> a -> [a]
dfs nexts found start  =  find start
  where
  find x  =  if found x then return x
             else do x' <- nexts x
                     find x'
```

The Haskell way of backtracking
Lazy evaluation produces only as many elements as you ask for.

14. Complexity and Optimization

Time complexity analysis
Optimizing functional programs

How to analyze and improve the time (and space) complexity of functional programs

Based largely on Richard Bird's book
Introduction to Functional Programming using Haskell.

Assumption in this section:

Reduction strategy is innermost (call by value, cbv)

- Analysis much easier
- Most languages follow cbv
- Number of lazy evaluation steps $\leq$ number of cbv steps
  $\Longrightarrow$ O-analysis under cbv also correct for Haskell
  but can be too pessimistic

14.1 Time complexity analysis

Basic assumption:

One reduction step takes one time unit

(No guards on the left-hand side of an equation,
`if-then-else` on the righ-hand side instead)

Justification:

The implementation does not copy data structures
but works with pointers and sharing

Example: `length (_ : xs) = length xs + 1`
Reduce `length [1,2,3]`

Compare:
```
id [] = []
id (x:xs) = x : id xs
```
Reduce `id [e1,e2]`
Copies list but shares elements.

$T_{\mathtt{f}}(n) =$ number of steps required for the evaluation of `f`
when applied to an argument of size n
in the worst case

What is “size”?

- Number of bits. Too low level.
- Better: specific measure based on the argument type of `f`
- Measure may differ from function to function.
- Frequent measure for functions on lists: the length of the list
  We use this measure unless stated otherwise
  Sufficient if `f` does not compute with the elements of the list
  Not sufficient for function . . .

How to calculate (not mechanically!) $T_{\mathtt{f}}(n)$:

1. From the equations for `f` derive equations for $T_{\mathtt{f}}$
2. If the equations for $T_{\mathtt{f}}$ are recursive, solve them

Example

```
[] ++ ys       =  ys
(x:xs) ++ ys   =  x : (xs ++ ys)
```

$$\begin{aligned} T_{++}(0, n) &= O(1) \\ T_{++}(m+1, n) &= T_{++}(m, n) + O(1) \end{aligned}$$

$$\Longrightarrow T_{++}(m, n) = O(m)$$

Note: `(++)` creates copy of first argument

Principle:

Every constructor of an algebraic data type takes time $O(1)$.
A constant amount of space needs to be allocated.

Example

```
reverse []      =  []
reverse (x:xs)  =  reverse xs ++ [x]
```

$$\begin{array}{lcl} T_{\texttt{reverse}}(0) & = & O(1) \\ T_{\texttt{reverse}}(n+1) & = & T_{\texttt{reverse}}(n) + T_{\texttt{++}}(n,1) \end{array}$$

$$\Longrightarrow T_{\texttt{++}}(n) = O(n^2)$$

Observation:

Complexity analysis may need functional properties
of the algorithm

The worst case time complexity of an expression e:

Sum up all $T_{\mathtt{f}}(n_1, ..., n_k)$
where $f\ e_1\ e_n$ is a function call in e
and n_i is the size of e_i

(assumption: no higher-order functions)

Note: examples so far equally correct with $\Theta(.)$ instead of $O(.)$, both for cbv and lazy evaluation. (Why?)

Consider `min xs = head(sort xs)`

$$T_{\mathtt{min}}(n) = T_{\mathtt{sort}}(n) + T_{\mathtt{head}}(n)$$

For cbv also a lower bound, but not for lazy evaluation.

Complexity analysis is *compositional* under cbv

14.2 Optimizing functional programs

> *Premature optimization is the root of all evil*
> *Don Knuth*

But we are in week $n-1$ now `;-)`

The ideal of program optimization:

1. Write (possibly) inefficient but correct code
2. Optimize your code *and prove equivelence to correct version*

Tail rercursion / Endrekursion

The definition of a function `f` is tail recursive / endrekursiv
if every recursive call is in "end position",
= it is the last function call before leaving `f`,
= nothing happens afterwards
= no call of `f` is not nested in another function call

Example

```
length []      =  0
length (x:xs)  =  length xs + 1

length2 []     n  =  n
length2 (x:xs) n  =  length2 xs (n+1)
```

```
length []      =  0
length (x:xs)  =  length xs + 1

length2 []     n  =  n
length2 (x:xs) n  =  length2 xs (n+1)
```

Compare executions:

```
length [a,b,c]
= length [b,c] + 1
= (length [c] + 1) + 1
= ((length [] + 1) + 1) + 1
= ((0 + 1) + 1) + 1
= 3

length2 [a,b,c] 0
= length2 [b,c] 1
= length2 [c]   2
= length2 []    3
= 3
```

Fact Tail recursive definitions can be compiled into loops.
Not just in functional languages.

No (additional) stack space is needed
to execute tail recursive functions

Example

```
length2 []     n  =  n
length2 (x:xs) n  =  length2 xs (n+1)
⇝
loop: if null xs then return n
      xs := tail xs
      n := n+1
      goto loop
```

Tail rercursion / Endrekursion

The definition of a function `f` is tail recursive / endrekursiv
if every recursive call is in “end position”,
= it is the last function call before leaving `f`,
= nothing happens afterwards
= no call of `f` is not nested in another function call

Example

```
length []       =  0
length (x:xs)   =  length xs + 1

length2 []     n  =  n
length2 (x:xs) n  =  length2 xs (n+1)
```

Accumulating parameters

An accumulating parameter is a parameter where intermediate results are accumulated.

Purpose:

- tail recursion
- replace `(++)` by `(:)`

```
length2 []     n  =  n
length2 (x:xs) n  =  length2 xs (n+1)

length' xs  =  length2 xs 0
```

Correctness:

Lemma `length2 xs n = length xs + n`

$\Longrightarrow$ `length' xs = length xs`

Tupling of results

Typical application:

Avoid multiple traversals of the same data structure

```
average :: [Float] -> Float
average xs = (sum xs) / (length xs)
```

Requires two traversals of the argument list.

Avoid intermediate data structures

Typical example: `map g . map f = map (g . f)`

Another example: `sum [n..m]`

Lazy evaluation

Not everything that is good for cbv is good for lazy evaluation

Problem: lazy evaluation may leave many expressions unevaluated until the end, which requires more space

Space is time because it requires garbage collection — not counted by number of reductions!