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Lambda Calculus
Winter Term 2017/18
Exercise Sheet 4

Exercise 1 (Confluence of β-Reduction)

In the lecture we have shown the confluence of $\longrightarrow_\beta$ using the diamond property of parallel β-reduction. In this exercise, we develop an alternative proof.

We define the operation $*$ on λ-terms inductively over the structure of terms:

$$\begin{array}{rcll} x^* & = & x & \\ (\lambda x.\, t)^* & = & \lambda x.\, t^* & \\ (t_1 t_2)^* & = & t_1^* t_2^* & \text{if } t_1 t_2 \text{ is not } \beta\text{-reducible.} \\ ((\lambda x.\, t_1) t_2)^* & = & t_1^*[t_2^*/x] & \end{array}$$

a) Show that we have for two arbitrary λ-terms s and t: $s > t \implies t > s^*$

b) Show that $\longrightarrow_\beta$ is confluent.

Exercise 2 (Parallel Beta Reduction)

Show:

$$s > t \Longrightarrow s \to_\beta^* t$$

Exercise 3 (Predecessor and Tail)

- Define a predecessor function pred on church numerals.
- Use the same idea to define tl on the list encoding from homework 2.5.

Homework 4 (Parallel Beta Reduction & Substitution)

Show:

$$s > s' \wedge t > t' \Longrightarrow s[t/x] > s'[t'/x]$$

Homework 5 (Equivalence modulo β-conversion)

Assume that we add the additional axiom

$$\lambda xy.x =_\beta \lambda xy.y$$

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a) Show that under this assumption $t =_\beta t'$ for all t, t'.

b) Repeat the same for the axiom $\lambda x.x =_\beta \lambda xy.yx$.

Homework 6 (Böhm's Theorem)

Böhm's Theorem states that for arbitrary closed terms $M \neq N$ without constant atoms in $\beta\eta$-normal form, there exist $n \geq 0$ and $L_1, \ldots, L_n$ such that:

$$M\ L_1\ \ldots\ L_n\ x\ y \to_\beta^* x \text{ and } N\ L_1\ \ldots\ L_n\ x\ y \to_\beta^* y\,.$$

That is, we can tell M and N apart. Show the following two special cases:

a) $M = \lambda xyz.xz(yz)$ and $N = \lambda xyz.x(yz)$

b) $M = \lambda xy.x(yy)$ and $N = \lambda xy.x(yx)$