Exercise 1 ($\beta$-reduction)

List all terms $t$ such that:

$$(\lambda x. (\lambda x y. x) z y) ((\lambda x x) (\lambda x x) ((\lambda x y. x) y)) \rightarrow^* \beta t$$

Which are normal forms?

Exercise 2 (Lists in $\lambda$-calculus)

Specify $\lambda$-terms for nil, cons, hd, tl and null, that encode lists in the $\lambda$-calculus. Show that your terms satisfy the following conditions:

- $\text{null } \text{nil} \rightarrow^* \beta \text{true}$
- $\text{hd } (\text{cons } x l) \rightarrow^* \beta x$
- $\text{null } (\text{cons } x l) \rightarrow^* \beta \text{false}$
- $\text{tl } (\text{cons } x l) \rightarrow^* \beta l$

Hint: Use pairs.
Homework 3 (Substitution Lemma)

Show that, given $x \neq y$ and $x \notin \text{FV}(u)$:

$$s[t/x][u/y] = s[u/y][t[u/y]/x]$$

Homework 4 (Trees in $\lambda$-calculus)

Encode a datatype of binary trees in lambda calculus. Specify the operations \texttt{tip} and \texttt{node} that construct trees, as well as \texttt{isTip}, \texttt{left}, \texttt{right}, and \texttt{value}. Each tip should carry a value, whereas each node should consist of two subtrees.

Show that the following holds:

- $\text{isTip} (\text{tip} a) \rightarrow^*_{\beta} \text{true}$
- $\text{isTip} (\text{node} x y) \rightarrow^*_{\beta} \text{false}$
- $\text{value} (\text{tip} a) \rightarrow^*_{\beta} a$
- $\text{left} (\text{node} x y) \rightarrow^*_{\beta} x$
- $\text{right} (\text{node} x y) \rightarrow^*_{\beta} y$

Homework 5 (Alternative Encoding of Lists)

In this exercise, we consider an alternative encoding of lists. The list $[x, y, z]$, for instance, will now be encoded as: $\lambda cn. cx (cy (cz n))$ (speaking in terms of functional programming, each list now encodes its corresponding \texttt{fold}). As in the tutorial, define the functions \texttt{nil}, \texttt{cons}, \texttt{hd}, and \texttt{null} for this encoding and show that they satisfy the same conditions. You do not need to define \texttt{tl}.

Homework 6 (Multiplication)

Define multiplication as a closed $\lambda$-term using \texttt{add} but no other helper functions and demonstrate its correctness on an example.