Exercise 1 (Confluence of $\beta$-Reduction)

In the lecture we have shown the confluence of $\rightarrow_{\beta}$ using the diamond property of parallel $\beta$-reduction. In this exercise, we develop an alternative proof.

We define the operation $\ast$ on $\lambda$-terms inductively over the structure of terms:

\[
\begin{align*}
x^\ast &= x \\
(\lambda x. t)^\ast &= \lambda x. t^\ast \\
(t_1 t_2)^\ast &= t_1^\ast t_2^\ast \quad \text{if } t_1 t_2 \text{ is not a $\beta$-redex.} \\
((\lambda x. t_1) t_2)^\ast &= t_1^\ast[t_2^\ast/x]
\end{align*}
\]

a) Show that we have for two arbitrary $\lambda$-terms $s$ and $t$: $s \tr\beta t \Rightarrow t \tr s^\ast$

b) Show that $\rightarrow_{\beta}$ is confluent.

Exercise 2 (Parallel Beta Reduction)

Show:

\[ s \tr\beta t \Rightarrow s \tr^\ast_{\beta} t \]

Exercise 3 (Predecessor and Tail)

a) Define a predecessor function $\text{pred}$ on church numerals.

b) Use the same idea to define $\text{tl}$ on the list encoding from homework 2.5.
Homework 4 (Parallel Beta Reduction & Substitution)

Show:
\[ s > s' \land t > t' \implies s[t/x] > s'[t'/x] \]

Homework 5 (Equivalence modulo $\beta$-conversion)

Assume that we add the additional axiom
\[ \lambda x. y. x =_\beta \lambda x. y \]

a) Show that under this assumption $t =_\beta t'$ for all $t$, $t'$.

b) Repeat the same for the axiom $\lambda x. y =_\beta \lambda x. y x$.

Homework 6 (Böhm’s Theorem)

Böhm’s Theorem states that for arbitrary closed terms $M \neq N$ without constant atoms in $\beta\eta$-normal form, there exist $n \geq 0$ and $L_1, \ldots, L_n$ such that:

\[ M L_1 \ldots L_n x y \rightarrow^* x \text{ and } N L_1 \ldots L_n x y \rightarrow^*_\beta y. \]

That is, we can tell $M$ and $N$ apart. Show the following two special cases:

a) $M = \lambda x. y z. x z (y z)$ and $N = \lambda x. y z. x (y z)$

b) $M = \lambda x. y. x (y y)$ and $N = \lambda x. y. x (y x)$