Exercise 1 (Intuitionistic Proofs)

Prove the following propositions in intuitionistic logic:

a) \((A \to A) \lor B\)

b) \(A \to (B \to A \land B)\)

c) \((A \to C) \to ((B \to C) \to (A \lor B \to C))\)

Exercise 2 (Intuitionistic Proof Search in Haskell)

The goal of this exercise is to implement the procedure to decide \(\Gamma \vdash A\) in Haskell, i.e., the algorithm from the proof of Theorem 4.1.2.

- Have a look at the template provided on the website. It provides definitions of formulae and proof terms of intuitionistic propositional logic.
- Try to fill in the implementation of `solve`.
- Implement the three proof rules seen in the lecture: `assumption`, `intro`, and `elim`. Use the examples at the end of the template to test your implementation as you go. For `elim`, use the criterion from the proof to guess suitable instantiations.

The algorithm can be streamlined further:

a) When trying to prove \(\Gamma \vdash A \to B\), it suffices to try \((\to\text{Intro})\). Explain why.

b) The attempt to prove \(\Gamma \vdash A\) by assumption can be dropped if we use the following generalised \(\to\text{Elim}\) rule:

\[
\frac{\Gamma \vdash A_1 \to \ldots \to A_n \to A \to B \quad \forall i \leq n. \quad \Gamma \vdash A_i}{\Gamma \vdash A \to B} \quad \to\text{ELIM}
\]

However, the proof obtained can be different. Explain the difference and why the outright proof by assumption is subsumed.
Homework 3 (From Proof Terms to Propositions)

Consider the following proof term:

\[ \lambda q. \lambda p. \text{case } \pi_1 p \text{ of } \text{in}_1 a \Rightarrow \text{in}_1 (\pi_1 q, (a, \pi_2 p)) | \text{in}_2 b \Rightarrow \text{in}_2 (\pi_2 q, b) \]

a) Exhibit the proposition that is proved by this term.

b) Give the corresponding proof tree.

Homework 4 (Intuitionistic Proofs)

Prove the following propositions in pure logic, without lambda-terms, and write down the \(\lambda\)-term corresponding to each proof:

a) \(\neg (A \lor B) \rightarrow \neg A \land \neg B\)

b) \(\neg A \land \neg B \rightarrow \neg (A \lor B)\)

Homework 5 (The Negative Fragment)

In this exercise, we consider the the fragment of intuitionistic logic where the only logical operator is \(\rightarrow\). We say that a formula \(A\) is negative if atomic formulas \(P\) only occur negated in \(A\), i.e. in the form \(P \rightarrow \bot\) (\(\neg P\) for short).

Show, by induction on \(A\), that if \(A\) is negative, then:

\[ \vdash \neg \neg \neg A \rightarrow A \]

Hint: First show:

a) \(\vdash \neg \neg \neg A \rightarrow \neg A\)

b) \(\vdash \neg \neg (A \rightarrow B) \rightarrow (\neg \neg A \rightarrow \neg \neg B)\)

c) \(\vdash (\neg \neg A \rightarrow \neg \neg B) \rightarrow (A \rightarrow \neg \neg B)\)