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Lambda Calculus
Winter Term 2021/22
Solutions to Exercise Sheet 10

Exercise 1 (Intuitionistic Proofs)

Prove the following propositions in intuitionistic logic:

a) $(A \to A) \lor B$

b) $A \to (B \to A \land B)$

c) $(A \to C) \to ((B \to C) \to (A \lor B \to C))$

Solution

a) Term: $\mathsf{in}_1\ (\lambda x.\ x)$. Proof:

$$\dfrac{\dfrac{\overline{A \vdash A}}{\vdash A \to A}\ {\to}\mathrm{I}}{\vdash (A \to A) \lor B}\ {\lor}\mathrm{I}_1$$

b) Term: $\lambda x\, y.\ \langle x, y \rangle$. Proof:

$$\dfrac{\dfrac{\dfrac{\overline{A, B \vdash A} \qquad \overline{A, B \vdash B}}{A, B \vdash A \land B}\ {\land}\mathrm{I}}{A \vdash B \to (A \land B)}\ {\to}\mathrm{I}}{\vdash A \to B \to (A \land B)}\ {\to}\mathrm{I}$$

c) Term: $\lambda x\, y\, z.\ \mathsf{case}\ z\ \mathsf{of}\ \mathsf{in}_1\ a \Rightarrow x\ a \mid \mathsf{in}_2\ b \Rightarrow y\ b$. Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma \vdash A \lor B} \qquad \dfrac{\overline{\Gamma, A \vdash A \to C} \qquad \overline{\Gamma, A \vdash A}}{\Gamma, A \vdash C}\ {\to}\mathrm{E} \qquad \dfrac{\overline{\Gamma, B \vdash B \to C} \qquad \overline{\Gamma, B \vdash B}}{\Gamma, B \vdash C}\ {\to}\mathrm{E}}{\Gamma := (A \to C), (B \to C), A \lor B \vdash C}\ {\lor}\mathrm{E}}{(A \to C), (B \to C) \vdash (A \lor B) \to C}\ {\to}\mathrm{I}}{(A \to C) \vdash (B \to C) \to (A \lor B) \to C}\ {\to}\mathrm{I}}{\vdash (A \to C) \to (B \to C) \to (A \lor B) \to C}\ {\to}\mathrm{I}$$

Exercise 2 (Intuitionistic Proof Search in Haskell)

The goal of this exercise is to implement the procedure to decide $\Gamma \vdash A$ in Haskell, i.e. the algorithm from the proof of Theorem 4.1.2.

- Have a look at the template provided on the website. It provides definitions of formulae and proof terms of intuitionistic propositional logic.
- Try to fill in the implementation of *solve*.
- Implement the three proof rules seen in the lecture: *assumption*, *intro*, and *elim*. Use the examples at the end of the template to test your implementation as you go. For *elim*, use the criterion from the proof to guess suitable instantiations.

The algorithm can be streamlined further:

a) When trying to prove $\Gamma \vdash A \to B$, it suffices to try ($\to$Intro). Explain why.

b) The attempt to prove $\Gamma \vdash A$ by assumption can be dropped if we use the following generalised $\to$Elim rule:

$$\frac{\Gamma \vdash A_1 \to \ldots \to A_n \to B \qquad \forall i \leq n.\ \ \Gamma \vdash A_i}{\Gamma \vdash B} \to\textsc{Elim}$$

However, the proof obtained can be different. Explain the difference and why the outright proof by assumption is subsumed.

Solution

See `prover.hs` for the implementation.

In the following we will denote the by ($\to$Elim) the more general rule described in lemma 4.1.2.

a) Suppose we prove $\Gamma \vdash A \to B$ by an application of ($\to$Elim). The proof will be of the following format:

$$\frac{\Gamma \vdash A_1 \to \ldots \to A_n \to A \to B \qquad \forall i \leq n.\ \ \Gamma \vdash A_i}{\Gamma \vdash A \to B} \to\textsc{Elim}$$

We can always provide an alternative proof that uses ($\to$Intro) first and looks like this:

$$\cfrac{\cfrac{\Gamma \vdash A_1 \to \ldots \to A_n \to A \to B \qquad \forall i.\ \ \Gamma, A \vdash A_i \qquad \Gamma, A \vdash A}{\Gamma, A \vdash B} \to\textsc{Elim}}{\Gamma \vdash A \to B} \to\textsc{Intro}$$

The case where $\Gamma \vdash A \to B$ is proved by assumption is subsumed by the next answer.

b) Proof by assumption is just a special case of ($\to$Elim) where $n = 0$. However, if we drop the assumption rule, proofs can now have a slightly different structure because we try ($\to$Intro) first:

$$\dfrac{\dfrac{A_1 \to \ldots \to A_n \to B \in \Gamma' \qquad \forall i \leq n.\ \ \Gamma' \vdash A_i}{\Gamma' \vdash B}\ \to\textsc{Elim}}{\Gamma, A_1 \to \ldots \to A_n \to B \vdash A_1 \to \ldots \to A_n \to B}\ \to\textsc{Intro}\ n\ \textsc{times}$$

with

$$\Gamma' := \Gamma, A_1 \to \ldots \to A_n \to B, A_1, \ldots, A_n\,.$$

Homework 3 (From Proof Terms to Propositions)

Consider the following proof term:

$$\lambda\, q.\ \lambda\, p.\ \ \mathsf{case}\ \pi_1\ p\ \mathsf{of}\ \mathsf{in}_1\ a \Rightarrow \mathsf{in}_1\ (\pi_1\ q,\ (a,\ \pi_2\ p))\ |\ \mathsf{in}_2\ b \Rightarrow \mathsf{in}_2\ (\pi_2\ q,\ b)$$

a) Exhibit the proposition that is proved by this term.

b) Give the corresponding proof tree.

Homework 4 (Intuitionistic Proofs)

Prove the following propositions in pure logic, without lambda-terms, and write down the λ-term corresponding to each proof:

a) $\neg(A \vee B) \rightarrow \neg A \wedge \neg B$

b) $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$

Homework 5 (The Negative Fragment)

In this exercise, we consider the the fragment of intuitionistic logic where the only logical operator is $\rightarrow$. We say that a formula A is negative if atomic formulas P only occur *negated* in A, i.e. in the form $P \rightarrow \bot$ ($\neg P$ for short).

Show, by induction on A, that if A is negative, then:

$$\vdash \neg\neg A \rightarrow A$$

Hint: First show:

a) $\vdash \neg\neg\neg A \rightarrow \neg A$

b) $\vdash \neg\neg(A \rightarrow B) \rightarrow (\neg\neg A \rightarrow \neg\neg B)$

c) $\vdash (\neg\neg A \rightarrow \neg\neg B) \rightarrow (A \rightarrow \neg\neg B)$