Exercise 1 (Intuitionistic Proofs)

Prove the following propositions in intuitionistic logic:

a) \((A \rightarrow A) \vee B\)

b) \(A \rightarrow (B \rightarrow A \land B)\)

c) \((A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \lor B \rightarrow C))\)

Solution

a) \(\text{in}_1 (\lambda x. \ x)\)

b) \(\lambda xy. \langle x, y \rangle\)

c) \(\lambda xyz. \ \text{case } z \text{ of } \text{in}_1 a \Rightarrow xa | \text{in}_2 b \Rightarrow yb\)

Exercise 2 (Intuitionistic Proof Search in Haskell)

The goal of this exercise is to implement the procedure to decide \(\Gamma \vdash A\) in Haskell, i.e. the algorithm from the proof of Theorem 4.1.2.

- Have a look at the template provided on the website. It provides definitions of formulae and proof terms of intuitionistic propositional logic.
- Try to fill in the implementation of solve.
- Implement the three proof rules seen in the lecture: assumption, intro, and elim. Use the examples at the end of the template to test your implementation as you go. For elim, use the criterion from the proof to guess suitable instantiations.

The algorithm can be streamlined further:

a) When trying to prove \(\Gamma \vdash A \rightarrow B\), it suffices to try \((\rightarrow\text{Intro})\). Explain why.

b) The attempt to prove \(\Gamma \vdash A\) by assumption can be dropped if we use the following generalised \(\rightarrow\text{Elim}\) rule:

\[
\begin{array}{c}
\Gamma \vdash A_1 \rightarrow \ldots \rightarrow A_n \rightarrow A \rightarrow B \\
\forall i \leq n. \ \Gamma \vdash A_i
\end{array}
\Rightarrow \frac{\Gamma \vdash A \rightarrow B}{\rightarrow\text{Elim}}
\]

However, the proof obtained can be different. Explain the difference and why the outright proof by assumption is subsumed.
Solution

See `prover.hs` for the implementation.

In the following we will denote the by (→Elim) the more general rule described in lemma 4.1.2.

a) Suppose we prove $\Gamma \vdash A \rightarrow B$ by an application of (→Elim). The proof will be of the following format:

$$
\frac{
\Gamma \vdash A_1 \rightarrow \cdots \rightarrow A_n \rightarrow A \rightarrow B \\
\forall i \leq n. \; \Gamma \vdash A_i
}{
\Gamma \vdash A \rightarrow B
} \rightarrow \text{ELIM}
$$

We can always provide an alternative proof that uses (→Intro) first and looks like this:

$$
\frac{
\Gamma \vdash A_1 \rightarrow \cdots \rightarrow A_n \rightarrow A \rightarrow B \\
\forall i. \; \Gamma, A \vdash A_i \\
\Gamma, A \vdash A
}{
\Gamma \vdash B
} \rightarrow \text{ELIM}
\frac{
\Gamma \vdash A \rightarrow B
}{
\Gamma \vdash \text{Intro} \; (n-1) \text{ times}
} \rightarrow \text{INTRO}
$$

The case where $\Gamma \vdash A \rightarrow B$ is proved by assumption is subsumed by the next answer.

b) Proof by assumption is just a special case of (→Elim) where $n = 0$. However, if we drop the assumption rule, proofs can now have a slightly different structure because we try (→Intro) first:

$$
\frac{
A_1 \rightarrow \cdots \rightarrow A_{n-1} \in \Gamma' \\
\forall i < n. \; \Gamma' \vdash A_i
}{
\Gamma' \vdash A_n
} \rightarrow \text{ELIM}
\frac{
\Gamma', A_1 \vdash \cdots \rightarrow A_n
}{
\Gamma, A_1 \vdash \cdots \rightarrow A_n
} \rightarrow \text{INTRO} \; (n-1) \text{ times}
$$

with

$$
\Gamma' := \Gamma, A_1 \rightarrow \cdots \rightarrow A_n, A_1, \ldots, A_{n-1}.
$$
Homework 3 (From Proof Terms to Propositions)

Consider the following proof term:

\[
\lambda q. \lambda p. \text{case } \pi_1 p \text{ of } \text{in}_1 a \Rightarrow \text{in}_1 (\pi_1 q, (a, \pi_2 p)) \mid \text{in}_2 b \Rightarrow \text{in}_2 (\pi_2 q, b)
\]

a) Exhibit the proposition that is proved by this term.

b) Give the corresponding proof tree.

Homework 4 (Intuitionistic Proofs)

Prove the following propositions in pure logic, without lambda-terms, and write down the \(\lambda\)-term corresponding to each proof:

a) \(\neg (A \lor B) \rightarrow \neg A \land \neg B\)

b) \(\neg A \land \neg B \rightarrow \neg (A \lor B)\)

Homework 5 (The Negative Fragment)

In this exercise, we consider the the fragment of intuitionistic logic where the only logical operator is \(\rightarrow\). We say that a formula \(A\) is negative if atomic formulas \(P\) only occur negated in \(A\), i.e. in the form \(P \rightarrow \bot\) (\(\neg P\) for short).

Show, by induction on \(A\), that if \(A\) is negative, then:

\[\vdash \neg \neg A \rightarrow A\]

*Hint:* First show:

a) \(\vdash \neg \neg A \rightarrow \neg A\)

b) \(\vdash \neg \neg (A \rightarrow B) \rightarrow (\neg \neg A \rightarrow \neg \neg B)\)

c) \(\vdash (\neg \neg A \rightarrow \neg \neg B) \rightarrow (A \rightarrow \neg \neg B)\)