Exercise 1 (Confluence of $\beta$-Reduction)

In the lecture we have shown the confluence of $\rightarrow_\beta$ using the diamond property of parallel $\beta$-reduction. In this exercise, we develop an alternative proof.

We define the operation $\ast$ on $\lambda$-terms inductively over the structure of terms:

$$
\begin{align*}
x^* &= x \\
(\lambda x. t^*) &= \lambda x. t^* \\
(t_1 t_2^*) &= t_1^* t_2^* \quad \text{if } t_1 t_2 \text{ is not a } \beta\text{-redex.} \\
((\lambda x. t_1) t_2^*) &= t_1^*[t_2^*/x]
\end{align*}
$$

a) Show that we have for two arbitrary $\lambda$-terms $s$ and $t$: $s \succ t \implies t \succ s^*$

b) Show that $\rightarrow_\beta$ is confluent.

Exercise 2 (Parallel Beta Reduction)

Show:

$$
s \succ t \implies s \rightarrow_\beta^* t
$$

Exercise 3 (Predecessor and Tail)

a) Define a predecessor function $\text{pred}$ on church numerals.

b) Use the same idea to define $\text{tl}$ on the fold encoding for lists.
Homework 4 (Parallel Beta Reduction & Substitution)

Show:

\[ s > s' \land t > t' \implies s[t/x] > s'[t'/x] \]

Homework 5 (Equivalence modulo \( \beta \)-conversion)

Assume that we add the additional axiom

\[ \lambda x\ y.\ x =_{\beta} \lambda x\ y.\ y \]

a) Show that under this assumption \( t =_{\beta} t' \) for all \( t, t' \).

b) Repeat the same for the axiom \( \lambda x.\ x =_{\beta} \lambda x\ y.\ y\ x \).

Homework 6 (Böhm’s Theorem)

Böhm’s Theorem states that for arbitrary closed terms \( M \neq N \) without constant atoms in \( \beta\eta \)-normal form, there exist \( n \geq 0 \) and \( L_1, \ldots, L_n \) such that:

\[ M \ L_1 \ldots L_n \ x \ y \rightarrow^{*}_{\beta} x \text{ and } N \ L_1 \ldots L_n \ x \ y \rightarrow^{*}_{\beta} y. \]

That is, we can tell \( M \) and \( N \) apart. Show the following two special cases:

a) \( M = \lambda x\ y\ z.\ x\ z\ (y\ z) \) and \( N = \lambda x\ y\ z.\ x\ (y\ z) \)

b) \( M = \lambda x\ y.\ x\ (y\ y) \) and \( N = \lambda x\ y.\ x\ (y\ x) \)