Exercise 1 (Peirce’s Law in Intuitionistic Logic)

Prove the following variant of Peirce’s Law in intuitionistic logic:

\[((\((P \rightarrow Q) \rightarrow P) \rightarrow P) \rightarrow Q) \rightarrow Q\]
Exercise 2 (Intuitionistic Proof Search in Haskell)

The goal of this exercise is to implement the procedure to decide $\Gamma \vdash A$ in Haskell, i.e. the algorithm from the proof of Theorem 4.1.2.

- Have a look at the template provided on the website. It provides definitions of formulae and proof terms of intuitionistic propositional logic.
- Try to fill in the implementation of `solve`.
- Implement the three proof rules seen in the lecture: `assumption`, `intro`, and `elim`. Use the examples at the end of the template to test your implementation as you go. For `elim`, use the criterion from the proof to guess suitable instantiations.

The algorithm can be streamlined further:

a) When trying to prove $\Gamma \vdash A \rightarrow B$, it suffices to try $(\rightarrow \text{Intro})$. Explain why.

b) The attempt to prove $\Gamma \vdash A$ by assumption can be dropped if we use the following generalised $\rightarrow \text{Elim}$ rule:

\[
\begin{array}{c}
\Gamma \vdash A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B \\
\forall i \leq n. \ \Gamma \vdash A_i
\end{array} \quad \rightarrow \text{Elim}
\]

However, the proof obtained can be different. Explain the difference and why the outright proof by assumption is subsumed.
Homework 3 (Constructive Logic)

a) Prove the following statement using the calculus for intuitionistic propositional logic:

\[((\mathbf{c} \rightarrow \mathbf{b}) \rightarrow \mathbf{c}) \rightarrow \mathbf{b} \rightarrow ((\mathbf{a} \rightarrow \mathbf{b}) \rightarrow \mathbf{b})\]

*Hint:* To make your proof tree more compact, you may remove unneeded assumptions to the left of the $\vdash$ during the proof as you see fit. For example, the following step is valid:

\[
\frac{p \vdash p}{p, q \vdash p}
\]

b) Give a well-typed expression in $\lambda^\rightarrow$ with the type

\[((\gamma \rightarrow \beta) \rightarrow \beta) \rightarrow (\gamma \rightarrow \alpha) \rightarrow ((\alpha \rightarrow \beta) \rightarrow \beta)\]

(You don’t need to give the derivation tree.)

Homework 4 (The Negative Fragment)

We say that a formula $A$ is negative if atomic formulas $P$ only occur negated in $A$, i.e. in the form $P \rightarrow \bot$ ($\neg P$ for short). The symbol $\bot$ for falsehood plays the role of an unprovable propositional constant: we do not have any special proof rules or axioms for it.

Show that if $A$ is negative, then:

$\vdash \neg\neg A \rightarrow A$

*Hint:* First show:

a) $\vdash \neg\neg\neg A \rightarrow \neg A$

b) $\vdash \neg\neg(A \rightarrow B) \rightarrow (\neg\neg A \rightarrow \neg\neg B)$

c) $\vdash (\neg\neg A \rightarrow \neg\neg B) \rightarrow (A \rightarrow \neg\neg B)$