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Lambda Calculus
Winter Term 2022/23
Solutions to Exercise Sheet 1

Exercise 1 (λ-Terms)

Rewrite the following terms such that they are completely parenthesized and conform to the grammar for the λ-calculus given in the lecture (without any shortcut notations).

a) $u\,x\,(y\,z)\,(\lambda v.\ v\,y)$

b) $(\lambda x\,y\,z.\ x\,z\,(y\,z))\,u\,v\,w$

Rewrite the following terms such there are as few parentheses as possible, and apply all shortcut notation from the lecture:

c) $((u\,(\lambda x.\ (v\,(w\,x))))\,x)$

d) $(((w\,(\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\,z)\,(y\,z))))))\,u)\,v)$

Evaluate the following substitutions:

e) $(\lambda y.\ x\,(\lambda x.\ x))\,[(\lambda y.\ x\,y)/x]$

f) $(y\,(\lambda v.\ x\,v))\,[(\lambda y.\ v\,y)/x]$

Solution

a) $(((u\,x)\,(y\,z))\,(\lambda v.\ (v\,y)))$

b) $((((\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\,z)\,(y\,z)))))\,u)\,v)\,w)$

c) $u\,(\lambda x.\ v\,(w\,x))\,x$

d) $w\,(\lambda x\,y\,z.\ x\,z\,(y\,z))\,u\,v$

e)

$$
\begin{aligned}
& (\lambda y.\ x\,(\lambda x.\ x))\,[(\lambda y.\ x\,y)/x] \\
= \; & (\lambda y.\ x[(\lambda y.\ x\,y)/x]\,(\lambda x.\ x)[(\lambda y.\ x\,y)/x]) \\
= \; & (\lambda y.\ (\lambda y.\ x\,y)\,(\lambda x.\ x)[(\lambda y.\ x\,y)/x]) \\
= \; & (\lambda y.\ (\lambda y.\ x\,y)\,(\lambda x.\ x))
\end{aligned}
$$

f)

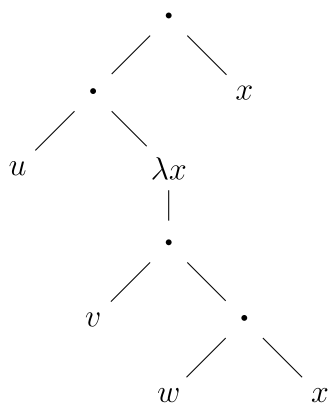
$$
\begin{aligned}
& (y\,(\lambda v.\ x\,v))\,[(\lambda y.\ v\,y)/x] \\
= \;& y[(\lambda y.\ v\,y)/x]\,(\lambda v.\ x\,v)[(\lambda y.\ v\,y)/x] \\
= \;& y\,(\lambda v.\ x\,v)[(\lambda y.\ v\,y)/x] \\
= \;& y\,(\lambda v'.\ (x\,v)[v'/v][(\lambda y.\ v\,y)/x]) \\
= \;& y\,(\lambda v'.\ (x[v'/v]\,v[v'/v])[(\lambda y.\ v\,y)/x]) \\
= \;& y\,(\lambda v'.\ (x\,v[v'/v])[(\lambda y.\ v\,y)/x]) \\
= \;& y\,(\lambda v'.\ (x\,v')[(\lambda y.\ v\,y)/x]) \\
= \;& y\,(\lambda v'.\ x[(\lambda y.\ v\,y)/x]\,v'[(\lambda y.\ v\,y)/x]) \\
= \;& y\,(\lambda v'.\ (\lambda y.\ v\,y)\,v'[(\lambda y.\ v\,y)/x]) \\
= \;& y\,(\lambda v'.\ (\lambda y.\ v\,y)\,v')
\end{aligned}
$$

Exercise 2 (λ-Terms as Trees)

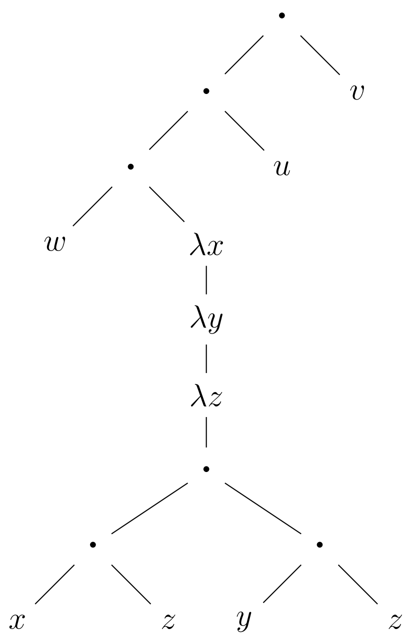
Rewrite the λ-terms resulting from exercises 1c) and 1d) to their corresponding representation as a tree.

Solution

a)



b)



Exercise 3 (Formalisations with λ-Terms)

Express the following propositions as λ-terms. Use the constant D as a derivative operator.

a) The derivative of x^2 is $2x$.

b) The derivative of x^2 at 3 is 6.

c) Let f be a function, and let g be defined as $g(x) := f(x^2)$. The derivative of g at x is different from the derivative of f at x^2.

d) Formulate the proposition c) without using the auxiliary function symbol g.

Solution

a) $(=)\,(D\,(\lambda x.\ (\hat{\ })\,x\,2))\,(\lambda x.\ (*)\,2\,x)$

b) $(=)\,(D\,(\lambda x.\ (\hat{\ })\,x\,2)\,3)\,6$

c) $(\neq)\,(D\,g\,x)\,(D\,f\,((\hat{\ })\,x\,2))$

d) $(\neq)\,(D\,(\lambda x.\ f\,((\hat{\ })\,x\,2))\,x)\,(D\,f\,((\hat{\ })\,x\,2))$

Homework 4 (Interpreting λ-Terms)

Give a *compact* natural-language description of the computational effect of the following λ-terms.

a) $\lambda x.\ x$

b) $\lambda x\, y.\ x$

c) $\lambda x\, y\, z.\ x\, z\, y$

d) $\lambda x\, y.\ x\,(x\, y)$

e) $\lambda x\, y\, z.\ x\,(y\, z)$

Homework 5 (Free and Bound Variables)

Mark the free variables in the following examples. Graphically indicate (by drawing arrows) the binding λ for each bound variable.

a) $\lambda x\, y\, z.\ (\lambda x\, y.\ z\, x)\, y\,(x\, z)$

b) $\lambda x.\ \lambda y.\ (\lambda y.\ z\,(\lambda z.\ y\, x))\,(\lambda x\, z.\ x\, y\, z)\, y\, x$

Homework 6 (Substitutions)

Evaluate the following substitutions:

a) $((\lambda x.\ f\, x)\,(\lambda f.\ f\, x))\,[g\, x/f]$

b) $(\lambda f.\ \lambda y.\ f\, x\, y)\,[f\, y/x]$

Homework 7 (Properties of Substitution)

Evaluate the following substitutions:

a) Give a counterexample for

$$s[t/x][u/y] = s[u/y][t/x].$$

b) Under which conditions does

$$s[t/x][u/y] = s[t[u/y]/x]$$

hold?