Exercise 1 (Confluence & Commutation)

Show: If $\rightarrow_1$ and $\rightarrow_2$ are confluent, and if $\rightarrow_1^*$ and $\rightarrow_2^*$ commute, then $\rightarrow_{12} := \rightarrow_1 \cup \rightarrow_2$ is also confluent.

Exercise 2 (Confluence of $\beta$-Reduction with Takahashi functions)

In the lecture, we have shown the confluence of $\rightarrow_\beta$ using the diamond property of parallel $\beta$-reduction. In this exercise, we develop an alternative proof based on what are sometimes called Takahashi functions. A function $\rho$ is a Takahashi function with respect to a reduction relation $>$ if it holds that

$$s > t \implies t > \rho(s).$$

a) Show that $\rightarrow$ is confluent if it holds that $\rightarrow \subseteq \rightarrow_\beta \subseteq \rightarrow^*$ and there exists a Takahashi function $\rho$ for $\rightarrow_\beta$.

We define the operation $-^*$ on $\lambda$-terms inductively over the structure of terms:

$$x^* = x$$
$$(\lambda x. t)^* = \lambda x. t^*$$
$$(t_1. t_2)^* = t_1^* t_2^*$$ if $t_1. t_2$ is not a $\beta$-redex.
$$((\lambda x. t_1) t_2)^* = t_1^*[t_2/x]$$

b) Show that $\rightarrow_\beta$ is confluent by proving that $-^*$ is a Takahashi function for the parallel and nested reduction $>$. 

Exercise 3 (Parallel Beta Reduction)

Show:

$$s > t \implies s \rightarrow_\beta^* t$$
Homework 4 (Local Confluence of \(\eta\)-reduction)

Analogously to \(\beta\)-reduction, we define \(\eta\)-reduction inductively:

1. \(x \notin \text{FV}(s) \implies (\lambda x. \ s \ x) \rightarrow_{\eta} s\)
2. \(s \rightarrow_{\eta} s' \implies s \ t \rightarrow_{\eta} s' \ t\)
3. \(s \rightarrow_{\eta} s' \implies t \ s \rightarrow_{\eta} t \ s'\)
4. \(s \rightarrow_{\eta} s' \implies (\lambda x. \ s) \rightarrow_{\eta} (\lambda x. \ s')\)

The proof of local confluence of \(\rightarrow_{\eta}\), i.e. it holds that there exists a \(u\) with \(t_1 \rightarrow_{\eta}^* u \leftarrow_{\eta} t_2\) if we have \(t_1 \leftarrow_{\eta} s \rightarrow_{\eta} t_2\), was very informal. Give a proper proof using this definition.

Homework 5 (Parallel Beta Reduction & Substitution)

Show:

\[ s > s' \land t > t' \implies s[t/x] > s'[t'/x]\]

Homework 6 (A Takahashi function for combinatory logic)

Instead of the \(\lambda\)-calculus, we consider combinatory logic in this exercise whose syntax consists of variables, application, and the combinators \(K\) and \(S\):

\[s, t ::= x \in \mathbb{N}_0 \mid s \mid t \mid K \mid S.\]

We inductively define a reduction relation \(\rightarrow_w\) for this calculus with:

1. \(K \ s \ t \rightarrow_w s\)
2. \(S \ s \ t \ u \rightarrow_w s \ u \ (t \ u)\)
3. \(s \rightarrow_w s' \implies s \ t \rightarrow_w s' \ t\)
4. \(t \rightarrow_w t' \implies s \ t \rightarrow_w s \ t'\)

Use the strategy from the tutorial to prove that \(\rightarrow_w\) is confluent by defining a parallel and nested reduction relation \(>_w\) for this calculus and a Takahashi function \(-^*\) for \(>_w\).