Exercise 1 (Normal Forms)

Recall the inductive definition of the set NF of normal forms:

\[
\begin{align*}
    t & \in \text{NF} \\
    \lambda x. t & \in \text{NF} \\
    n \geq 0 & \\
    t_1 \in \text{NF} & \\
    t_2 \in \text{NF} & \\
    \cdots & \\
    t_n \in \text{NF} & \\
    x \ t_1 \ t_2 \ \cdots \ t_n & \in \text{NF}
\end{align*}
\]

Show that this set precisely captures all normal forms, i.e.:

\[
t \in \text{NF} \iff \nexists t'. t \to_{\beta} t'
\]

Solution

Direction \(\implies\): Proof by induction on the derivation of \(t \in \text{NF}\).

Case \(t = \lambda x. s \in \text{NF}\)

We obtain the induction hypothesis \(s \in \text{NF} \implies \nexists t'. s \to_{\beta} t'\). Suppose there exists an \(s'\) such that \(\lambda x. s \to_{\beta} \lambda x. s'\). By rule inversion, it must hold that \(s \to_{\beta} s'\) which is a contradiction to induction hypothesis where we discharge the assumption \(s \in \text{NF}\) by rule inversion on \(t \in \text{NF}\).

Case \(t = x \ t_1 \ \cdots \ t_n \in \text{NF}\)

By rule inversion on \(t \in \text{NF}\) we get \(t_i \in \text{NF}\) and therefore \(\nexists t'. t_i \to_{\beta} t'\) by the IH for \(1 \leq i \leq n\). We show this case by another induction on \(n\). The case \(n = 0\) is trivial. Now assume \(\nexists t'. x \ t_1 \ t_2 \ \cdots \ t_n \to_{\beta} t'\) as the induction hypothesis, and \(x \ t_1 \ t_2 \ \cdots \ t_n \ t_{n+1} \to_{\beta} t'\) for the sake of contradiction. By analysing the derivation of the latter, we can only conclude \(\exists t'. t_{n+1} \to_{\beta} t'\), which manifests a contradiction.

Direction \(\impliedby\): We show \(t \notin \text{NF} \implies \exists t'. t \to_{\beta} t'\) by structural induction on \(t\).

Case \(t = x\)

Then \(t \in \text{NF}\) which is a contradiction to the assumption \(t \notin \text{NF}\).

Case \(t = \lambda x. s\)

We assume that \(t = \lambda x. s \notin \text{NF}\). Thus, it must hold that \(s \notin \text{NF}\) by rule inversion, which together with the IH lets us obtain a \(s'\) where \(s \to_{\beta} s'\). Hence, we conclude that \(\lambda x. s \to_{\beta} \lambda x. s'\).
**Case** $t = t_1 t_2$
We assume $t_1 \not\in \text{NF} \implies \exists t'. t_1 \to_{\beta} t'$ and $t_2 \not\in \text{NF} \implies \exists t'. t_2 \to_{\beta} t'$ as the induction hypothesis, and $t \not\in \text{NF}$.

The cases where $t_1 \not\in \text{NF}$ or $t_2 \not\in \text{NF}$ are immediate by the induction hypotheses together with the definition of $\to_{\beta}$.

Consider the case $t_1, t_2 \in \text{NF}$. We analyze the derivation of $t_1 \in \text{NF}$.

**Case** $t_1 = x s_1 \ldots s_n$ where $s_1, \ldots, s_n \in \text{NF}$
It follows that $t = t_1 t_2 = x s_1 \ldots s_n t_2 \in \text{NF}$ since $t_2 \in \text{NF}$. But this is a contradiction to the assumption $t_1 t_2 \not\in \text{NF}$.

**Case** $t_1 = (\lambda x. s)$ where $s \in \text{NF}$
Then it holds that $t = t_1 t_2 = (\lambda x. s) t_2 \to_{\beta} s[t_2/x]$.

**Exercise 2 (Weakly Normalising Terms)**
Inductively define the set of weakly normalising terms $\text{WN}$, i.e. the set of terms that have a $\beta$-normal form. In particular it should hold that

$$s \in \text{WN} \iff \exists t. s \Rightarrow^* t.$$

Similarly, define the set of strongly normalising terms $\text{SN}$ where a term $s$ is strongly normalising if there is no infinite sequence $\{t_i \mid i \in \mathbb{N}\}$ with $s \to^*_{\beta} t_0$ and $t_i \to_{\beta} t_{i+1}$ for $i \in \mathbb{N}$.

Give a term $t$ that is weakly but not strongly normalising.

**Solution**
We define $\text{WN}$ as follows:

1. $t_1, \ldots, t_m \in \text{WN} \implies x t_1 \ldots t_m \in \text{WN}$
2. $t \in \text{WN} \implies \lambda x. t \in \text{WN}$
3. $s[t/x] t_1 \ldots t_m \in \text{WN} \implies (\lambda x. s) t t_1 \ldots t_m \in \text{WN}$

The definition of $\text{SN}$ is identical with exception of the last rule where we also demand that $t \in \text{SN}$.

Let $\omega := (\lambda x. x x)$. Then, it holds that $(\lambda x. y) (\omega \omega) \in \text{WN} \setminus \text{SN}$.
Homework 3 (Normal Forms & Big Step)

Show:
\[ t \in \text{NF} \land t \Rightarrow_n u \Rightarrow u = t \]

Homework 4 (Characterisation of \( \text{WN} \))

Prove the following characterisation of \( \text{WN} \):
\[ t \in \text{WN} \iff \exists t'. t \rightarrow^* \beta t' \land t' \in \text{NF} \]

You may use the fact that
\[ s \in \text{WN} \iff \exists t. s \Rightarrow_n t \]

and all theorems up to Theorem 1.5.8 from the lecture.