First-Order Logic

Compactness

[Harrison, Section 3.16]
Recall Gödel-Herbrand-Skolem:

**Theorem**

Let $F$ be a closed formula in Skolem form. Then $F$ is satisfiable iff its Herbrand expansion $E(F)$ is (propositionally) satisfiable.

Can easily be generalized:

**Theorem (1)**

Let $S$ be a set of closed formulas in Skolem form. Then $S$ is satisfiable iff $E(S)$ is (propositionally) satisfiable.
Transforming sets of formulas

Recall the transformation of single formulas into equisatisfiable Skolem form: close, RPF, skolemize

Theorem (2)
Let $S$ be a countable set of closed formulas. Then we can transform it into an equisatisfiable set $T$ of closed formulas in Skolem form.
We call this transformation function skolem.

- Can all formulas in $S$ be transformed in parallel?
- Why countable?
Transforming sets of formulas

1. Put all formulas in $S$ into RPF.
   
   Problem in Skolemization step: How do we generate new function symbols if all of them have been used already in $S$?

2. Rename all function symbols in $S$: $f^k_i \mapsto f^k_{2i}$
   
   The result: equisatisfiable countable set $\{F_0, F_1, \ldots \}$.

   Unused symbols: all $f^k_{2i+1}$

3. Skolemize the $F_i$ one by one using the $f^k_{2i+1}$ not used in the Skolemization of $F_0, \ldots, F_{i-1}$

   Result is equisatisfiable with initial $S$. 

Compactness

**Theorem**

*Let $S$ be a countable set of closed formulas. If every finite subset of $S$ is satisfiable, then $S$ is satisfiable.*

**Proof**

every fin. $F \subseteq S$ is sat.

$\Rightarrow$ every fin. $F \subseteq \text{skolem}(S)$ is sat. by Theorem (2)

$\quad$ (fin. $F \subseteq \text{skolem}(S) \Rightarrow F \subseteq \text{skolem}(S_0)$ for some fin. $S_0 \subseteq S$)

$\Rightarrow$ for every fin. $F \subseteq \text{skolem}(S)$, $E(F)$ is prop. sat. by Theorem(1)

$\Rightarrow$ every fin. $F' \subseteq E(\text{skolem}(S))$ is prop. sat.

$\quad$ (there must exist a fin. $F \subseteq \text{skolem}(S)$ s.t. $F' \subseteq E(F)$)

$\Rightarrow E(\text{skolem}(S))$ is prop. sat. by prop. compactness

$\Rightarrow \text{skolem}(S)$ is sat. by Theorem (1)

$\Rightarrow S$ is sat. by Theorem (2)