Propositional Logic
Definitional CNF
Definitional CNF

The **definitional CNF** of a formula is obtained in 2 steps:

1. Repeatedly replace a subformula $G$ of the form $\neg A'$, $A' \land B'$ or $A' \lor B'$ by a new atom $A$ and conjoin $A \leftrightarrow G$. This replacement is not applied to the “definitions” $A \leftrightarrow G$ but only to the (remains of the) original formula.

2. Translate all the subformulas $A \leftrightarrow G$ into CNF.

**Example**

\[
\neg(A_1 \lor A_2) \land A_3
\]

\[
\Rightarrow
\]

\[
\neg A_4 \land A_3 \land (A_4 \leftrightarrow (A_1 \lor A_2))
\]

\[
\Rightarrow
\]

\[
A_5 \land A_3 \land (A_4 \leftrightarrow (A_1 \lor A_2)) \land (A_5 \leftrightarrow \neg A_4)
\]

\[
\Rightarrow
\]

\[
A_5 \land A_3 \land \text{CNF}(A_4 \leftrightarrow (A_1 \lor A_2)) \land \text{CNF}(A_5 \leftrightarrow \neg A_4)
\]
Definitional CNF: Complexity

Let the initial formula have size $n$.

1. Each replacement step increases the size of the formula by a constant.
   There are at most as many replacement steps as subformulas, linearly many.

2. The conversion of each $A \leftrightarrow G$ into CNF increases the size by a constant.
   There are only linearly many such subformulas.

Thus the definitional CNF has size $O(n)$. 
Notation

Definition
The notation $F[G/A]$ denotes the result of replacing all occurrences of the atom $A$ in $F$ by $G$. We pronounce it as “$F$ with $G$ for $A$”.

Example
$(A \land B)[(A \rightarrow B)/B] = (A \land (A \rightarrow B))$

Definition
The notation $\mathcal{A}[\nu/A]$ denotes a modified version of $\mathcal{A}$ that maps $A$ to $\nu$ and behaves like $\mathcal{A}$ otherwise:

$$(\mathcal{A}[\nu/A])(A_i) = \begin{cases} 
\nu & \text{if } A_i = A \\
\mathcal{A}(A_i) & \text{otherwise}
\end{cases}$$
Substitution Lemma

Lemma
\[ \mathcal{A}(F[G/A]) = \mathcal{A}'(F) \quad \text{where} \quad \mathcal{A}' = \mathcal{A}[\mathcal{A}(G)/A] \]

Example
\[ \mathcal{A}((A_1 \land A_2)[G/A_2]) = \mathcal{A}'(A_1 \land A_2) \quad \text{where} \quad \mathcal{A}' = \mathcal{A}[\mathcal{A}(G)/A_2] \]

Proof by structural induction on \( F \).

Case \( F \) is an atom:
- If \( F = A \):
  \[ \mathcal{A}(F[G/A]) = \mathcal{A}(G) = \mathcal{A}'(F) \]
- If \( F \neq A \):
  \[ \mathcal{A}(F[G/A]) = \mathcal{A}(F) = \mathcal{A}'(F) \]

Case \( F = F_1 \land F_2 \):
\[
\begin{align*}
\mathcal{A}(F[G/A]) &= \\
\mathcal{A}(F_1[G/A] \land F_2[G/A]) &= \\
\min(\mathcal{A}(F_1[G/A]), \mathcal{A}(F_2[G/A])) &\overset{IH}{=} \\
\min(\mathcal{A}'(F_1), \mathcal{A}'(F_2)) &= \mathcal{A}'(F_1 \land F_2) = \mathcal{A}'(F)
\end{align*}
\]
Definitional CNF: Correctness

Each replacement step produces an equisatisfiable formula:

Lemma

Let $A$ be an atom that does not occur in $G$. Then $F[G/A]$ is equisatisfiable with $F \land (A \leftrightarrow G)$.

Proof If $F[G/A]$ is satisfiable by some assignment $\mathcal{A}$, then by the Substitution Lemma, $\mathcal{A}' = \mathcal{A}[\mathcal{A}(G)/A]$ is a model of $F$. Moreover $\mathcal{A}' \models (A \leftrightarrow G)$ because $\mathcal{A}'(A) = \mathcal{A}(G)$ and $\mathcal{A}(G) = \mathcal{A}'(G)$ by the Coincidence Lemma (Exercise 1.2). Thus $F \land (A \leftrightarrow G)$ is satisfiable (by $\mathcal{A}'$).

Conversely we actually have $F \land (A \leftrightarrow G) \models F[G/A]$.

Suppose $\mathcal{A} \models F \land (A \leftrightarrow G)$. This implies $\mathcal{A}(A) = \mathcal{A}(G)$. Therefore $\mathcal{A}[\mathcal{A}(G)/A] = \mathcal{A}$.

Thus $\mathcal{A}(F[G/A]) = (\mathcal{A}[\mathcal{A}(G)/A])(F) = \mathcal{A}(F) = 1$ by the Substitution Lemma.

Does $F[G/A] \models F \land (A \leftrightarrow G)$ hold?
Summary

Theorem
For every formula $F$ of size $n$ there is an **equisatisfiable CNF** formula $G$ of size $O(n)$.

Similarly it can be shown:

Theorem
For every formula $F$ of size $n$ there is an **equivalent DNF** formula $G$ of size $O(n)$.
Validity of CNF

Validity of formulas in CNF can be checked in linear time.

A formula in CNF is valid iff all its disjunctions are valid.

A disjunction is valid iff it contains both an atomic $A$ and $\neg A$ as literals.

Example

Valid: $(A \lor \neg A \lor B) \land (C \lor \neg C)$

Not valid: $(A \lor \neg A) \land (\neg A \lor C)$
Satisfiability of formulas in DNF can be checked in linear time. A formula in DNF is satisfiable iff at least one of its conjunctions is satisfiable. A conjunction is satisfiable iff it does not contain both an atomic $A$ and $\neg A$ as literals.

**Example**
Satisfiable: $(\neg B \land A \land B) \lor (\neg A \land C)$
Unsatisfiable: $(A \land \neg A \land B) \lor (C \land \neg C)$
Satisfiability/validity of DNF and CNF

Theorem

*Satisfiability of formulas in CNF is NP-complete.*

Theorem

*Validity of formulas in DNF is co-NP-complete.*

The standard decision procedure for validity of $F$:

1. Transform $\neg F$ into an equisat. formula $G$ in def. CNF
2. Apply efficient CNF-based SAT solver to $G$