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**Exercise 3.1. [System G1c]**
An alternative definition of the sequent calculus ("G1c") is defined as follows, where $A, B$ are formulas:

*Axioms*

$$\text{Ax}\ A \Rightarrow A \qquad \text{L}\bot\ \bot \Rightarrow$$

*Rules for weakening (W) and contraction (C)*

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$\text{LC}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{RC}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

*Rules for the logical operators*

$$\text{L}\wedge\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\ (i = 0, 1) \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\ (i = 0, 1)$$

$$\text{L}\to\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \text{R}\to\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

Notably, weaking and contraction are built-in rules. Moreover, for system G1c, we define $\neg F := F \to \bot$.

Show that sequent calculus can be simulated by G1c, i.e., $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_{G1c} \Gamma \Rightarrow \Delta$.

**Solution:**
The rules $Ax$ and $\bot L$ are simulated by $Ax$ and $L\bot$ together with the weakening rules (by induction on $|\Gamma| + |\Delta|$).

We consider two rules for logical operators: $\wedge L$ and $\neg R$. The other cases are similar. We show how those can be simulated in G1c.

$$\text{LC}\ \dfrac{L\wedge\ \dfrac{L\wedge\ \dfrac{\mathbf{F}, \mathbf{G}, \boldsymbol{\Gamma} \Rightarrow \boldsymbol{\Delta}}{F, F \wedge G, \Gamma \Rightarrow \Delta}}{F \wedge G, F \wedge G, \Gamma \Rightarrow \Delta}}{\mathbf{F} \wedge \mathbf{G}, \boldsymbol{\Gamma} \Rightarrow \boldsymbol{\Delta}} \qquad \dfrac{R \to\ \dfrac{\text{RW}\ \dfrac{\mathbf{F}, \boldsymbol{\Gamma} \Rightarrow \boldsymbol{\Delta}}{F, \Gamma \Rightarrow \bot, \Delta}}{\Gamma \Rightarrow F \to \bot, \Delta}}{\boldsymbol{\Gamma} \Rightarrow \neg \mathbf{F}, \boldsymbol{\Delta}}$$

**Exercise 3.2.** **[Cut Elimination, Semantically]**
Semantically prove the admissibility of the following rule:

$$\text{If } \vdash_G \Gamma \Rightarrow F, \Delta \text{ and } \vdash_G F, \Gamma \Rightarrow \Delta \text{ then } \vdash_G \Gamma \Rightarrow \Delta$$

**Solution:**
To prove this semantically, we have to show that given $|\Gamma \Rightarrow F, \Delta|$ and $|F, \Gamma \Rightarrow \Delta|$, $|\Gamma \Rightarrow \Delta|$ holds. In fact, an even stronger property holds: precedent and antecedent are equivalent.

Assume $\models A \to (L \vee C)$ (1) and $\models L \wedge A \to C$ (2). We have to show that $\models A \to C$. Pick an assignment $\mathcal{A}$ with $\mathcal{A}(A) = 1$ (3). Then by (1) we have $\mathcal{A}(L \vee C) = 1$. Hence, either $\mathcal{A}(L) = 1$ or $\mathcal{A}(C) = 1$. In the latter case, we are done. In the former case, we obtain $\mathcal{A}(C)$ with (2,3).

Note: proving the statement for fixed $A, L, C$ suffices due to the substitution lemma (think of $A$, $L$, $C$ as metavariables you can instantiate if it helps you).

**Exercise 3.3.** **[More Connectives]**
Define sequent rules for the logical connectives "nand" ($\overline{\wedge}$) and "xor" ($\otimes$).

**Solution:**
The simplest way to derive the sequent rules is to consider the definition of $\overline{\wedge}$ and $\otimes$.

$$\begin{aligned} F \,\overline{\wedge}\, G &\equiv \neg(F \wedge G) \\ F \otimes G &\equiv (F \wedge \neg G) \vee (\neg F \wedge G) \end{aligned}$$

One can apply the sequent calculus rules on these definitions and simplify accordingly to obtain:

$$\overline{\wedge}L\ \frac{\Gamma \Rightarrow \Delta, F \qquad \Gamma \Rightarrow \Delta, G}{\Gamma, F \,\overline{\wedge}\, G \Rightarrow \Delta} \qquad \overline{\wedge}R\ \frac{\Gamma, F, G \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, F \,\overline{\wedge}\, G}$$

$$\otimes L\ \frac{\Gamma, F \Rightarrow \Delta, G \qquad \Gamma, G \Rightarrow \Delta, F}{\Gamma, F \otimes G \Rightarrow \Delta} \qquad \otimes R\ \frac{\Gamma \Rightarrow \Delta, F, G \qquad \Gamma, F, G \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, F \otimes G}$$

**Exercise 3.4.** **[Inversion]**
Show that the following inversion rule is admissible using proof transformations:

$$\frac{\Gamma \Rightarrow F \vee G, \Delta}{\Gamma \Rightarrow F, G, \Delta}$$

**Solution:**
Proof by induction on the depth of the premise's proof tree, denoted by $n$. The base case is vacuous because there are no proof trees of depth 0. In case $n + 1$, proceed by case analysis on the final rule of the premise's proof tree:

The cases $\bot L$ and $Ax$ are trivial (you should ask yourself "why?" and be able to answer it).

Now if $F \vee G$ is the principal formula, then the only applicable rule is $\vee R$. Hence, as the premise of $\vee R$, we obtain a derivation $\Gamma \Rightarrow_n F, G, \Delta$ and we are done.

If $F \vee G$ is not the principal formula, in all cases, we apply the inductive hypotheses to all proof trees of the rule's assumptions and then reapply the rule to the obtained proofs to construct our final derivation (cf lecture).

Here is the case for $\wedge R$: We have $\Delta = F' \wedge G', \Delta'$ and

$$\frac{\Gamma \Rightarrow_n F \vee G, F', \Delta' \qquad \Gamma \Rightarrow_n F \vee G, G', \Delta'}{\Gamma \Rightarrow_{n+1} F \vee G, F' \wedge G', \Delta'} \ (\wedge R)$$

by assumption. Thus, by the IH, we obtain $\Gamma \Rightarrow_n F, G, F', \Delta'$ and $\Gamma \Rightarrow_n F, G, G', \Delta'$. Applying $\wedge R$ to those proofs finishes the proof:

$$\frac{\Gamma \Rightarrow_n F, G, F', \Delta' \qquad \Gamma \Rightarrow_n F, G, G', \Delta'}{\Gamma \Rightarrow_{n+1} F, G, F' \wedge G', \Delta'} \ (\wedge R)$$

**Homework 3.1.** **[Stay Classy]** (++)

1. Prove the formulas $F \vee \neg F$ (tertium non datur) and $(\neg F \to F) \to F$ (consequentia mirabilis) in System G1c.
2. The intuitionistic system "G1l" is the subsystem of G1c obtained by restricting all rules to sequents with at most one succedent formula and by replacing the rule $L\!\to$ with

$$\frac{\Gamma \Rightarrow F \qquad G, \Gamma \Rightarrow H}{F \to G, \Gamma \Rightarrow H}$$

   Can you prove either of former formulas in G1l? How about their doubly negated forms $\neg\neg(F \vee \neg F)$ and $\neg\neg((\neg F \to F) \to F)$?

**Solution:**

1. The proofs are (almost) syntax-directed. Here's the proof of the former:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{F \Rightarrow F, \bot}(RW + Ax)}{\Rightarrow F, \neg F}(R\!\to)}{\Rightarrow F \vee \neg F, F \vee \neg F}(2 * R\vee)}{\Rightarrow F \vee \neg F}(RC)$$

Bonus note: in both proofs, we include a step in which we prove $F, \neg F$ by assuming $F$ (the assumption of $\neg F \equiv F \to \bot$) and then use this $F$ to show the former disjunct of the goal. This proof step can be given a computational interpretation using continuations: when faced with a proof of $F$ or $\neg F$, the computer simply assumes $\neg F \equiv F \to \bot$ holds. But if you then ever want to make use of $\neg F$ by passing it a proof of $F$, it will change its mind and jump back in time to instead tell you that $F$ has always been true and hand you back the proof you just passed. Here is a related Faustian bargain story by Philip Wadler:

> The devil: Here is my offer. Either (a) I will give you one million euros, or (b) I will grant you any wish if you pay me one million euros.
>
> Faust: No other conditions? Do I need to sign over my soul?
>
> The devil: Keep it. But I get to choose whether I offer (a) or (b).
>
> Faust: I accept. Do I get (a) or (b)?
>
> The devil: I choose (b).
>
> Many years later, Faust returns with one million euros and gives it to the devil.
>
> Faust: Grant me my wish!
>
> The devil: Oh, did I say (b) before? I'm so sorry. I meant (a). It is my great pleasure to give you one million euros.

2. Both formulas are not derivable in G1l.[1] However, their doubly negated forms indeed are. Here's the proof for the latter formula: Let $G :\equiv (\neg F \to F) \to F$, then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg G \Rightarrow G}(*) \qquad \dfrac{}{\neg G, \bot \Rightarrow \bot}(LW + L\bot)}{\neg G, \neg G \Rightarrow \bot}(L{\to})}{\neg G \Rightarrow \bot}(LC)}{\Rightarrow \neg\neg G}(R{\to})$$

where (*) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F, \neg F \to F \Rightarrow F}(LW + Ax)}{F \Rightarrow G}(R{\to}) \qquad \dfrac{}{F, \bot \Rightarrow \bot}(LW + Ax)}{\neg G, F \Rightarrow \bot}(L{\to})}{\neg G \Rightarrow \neg F}(R{\to}) \qquad \dfrac{}{\neg G, F \Rightarrow F}(LW + Ax)}{\neg G, \neg F \to F \Rightarrow F}(L{\to})}{\neg G \Rightarrow G}(R{\to})$$

**Homework 3.2. [Inversion Rules]** (++)

Show that the following inversion rules are admissible using proof transformations:

$$\dfrac{F \wedge G, \Gamma \Rightarrow \Delta}{F, G, \Gamma \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow F \to G, \Delta}{F, \Gamma \Rightarrow G, \Delta}$$

**Solution:**

The proofs are by induction on the depth of the premise's proof tree. They are analagous to the inversion lemma proofs done in the lecture and in the tutorial.

**Homework 3.3. [Sequent Prover]** (+++)

Implement a sequent calculus prover in a high-level programming language, and test it for examples from this exercise sheet, the lecture, or your own.

> If I had a world of my own, everything would be nonsense.
>
> — Alice

[1]For those of you that stayed during our tutorial follow-up discussions: you can show this, for example, by constructing a Kripke model refuting the formula(s).